New reductions of integrable matrix PDEs —$Sp(m)$-invariant systems—

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Abstract

We propose a new type of reduction for integrable systems of coupled matrix PDEs; this reduction equates one matrix variable with the transposition of another multiplied by an *antisymmetric* constant matrix. Via this reduction, we obtain a new integrable system of coupled derivative mKdV equations and a new integrable variant of the massive Thirring model, in addition to the already known systems. We also discuss integrable semi-discretizations of the obtained systems and present new soliton solutions to both continuous and semi-discrete systems. As a by-product, a new integrable semi-discretization of the Manakov model (self-focusing vector NLS equation) is obtained.

Keywords: $Sp(m)$-invariant systems, matrix derivative NLS hierarchies, matrix Yajima–Oikawa hierarchy, coupled derivative mKdV equations, massive Thirring model, *potential Kaup–Newell equation*, integrable discretizations, discrete Kaup–Newell system, Gel'fand–Levitan–Marchenko integral equations, soliton solutions

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1 Introduction

Since the seminal work of Manakov [1] in the early 70s, integrable systems of coupled partial differential equations (PDEs) that are associated with higher than second-order matrix spectral problems (Lax pairs) have been the focus of intensive research. In particular, of prime importance among such systems are the vector PDEs that are invariant under the action of a classical matrix group on the vector dependent variables; this invariance also represents a "symmetry" or gauge invariance of the spectral problem. Because of this large "symmetry", the vector PDEs usually allow the existence of solitons with internal degrees of freedom that exhibit highly nontrivial and interesting behaviors in soliton interactions. Moreover, their simple and symmetric form of equations very often leads to their potential or practical applicability in various branches of physics as well as applied mathematics. Typical examples of such vector PDEs include the $U(m)$-invariant Manakov model [1, 2], also referred to as the self-focusing vector nonlinear Schrödinger (NLS) equation,

$$\mathrm{i}\boldsymbol{q}_t + \boldsymbol{q}_{xx} + 2\|\boldsymbol{q}\|^2\boldsymbol{q} = \boldsymbol{0}, \quad \boldsymbol{q} = (q_1, q_2, \dots, q_m), \quad \|\boldsymbol{q}\|^2 := \boldsymbol{q}\cdot\boldsymbol{q}^\dagger = \sum_{j=1}^{m} |q_j|^2; \tag{1.1}$$

two distinct versions of the vector mKdV equation [3–5] with $O(m)$-invariance,

$$\begin{aligned} &\boldsymbol{q}_t + \boldsymbol{q}_{xxx} + 6\langle\boldsymbol{q},\boldsymbol{q}\rangle\boldsymbol{q}_x = \boldsymbol{0}, \quad \boldsymbol{q} = (q_1, q_2, \dots, q_m), \quad \langle\boldsymbol{q},\boldsymbol{q}\rangle := \boldsymbol{q}\cdot\boldsymbol{q}^T = \sum_{j=1}^{m} q_j^2, \\ &\boldsymbol{q}_t + \boldsymbol{q}_{xxx} + 3\langle\boldsymbol{q},\boldsymbol{q}_x\rangle\boldsymbol{q} + 3\langle\boldsymbol{q},\boldsymbol{q}\rangle\boldsymbol{q}_x = \boldsymbol{0}; \end{aligned} \tag{1.2}$$

the vector third-order Heisenberg ferromagnet model with $O(m)$-invariance [6, 7],

$$\boldsymbol{S}_t + \boldsymbol{S}_{xxx} + \frac{3}{2}(\langle\boldsymbol{S}_x,\boldsymbol{S}_x\rangle\boldsymbol{S})_x = \boldsymbol{0}, \quad \langle\boldsymbol{S},\boldsymbol{S}\rangle = 1; \tag{1.3}$$

and the $O(m)$-invariant vector extension [7] of the third-order Wadati–Konno–Ichikawa equation [8],

$$\boldsymbol{q}_t + \left[\frac{\boldsymbol{q}_x}{(1-\langle\boldsymbol{q},\boldsymbol{q}\rangle)^{\frac{3}{2}}}\right]_{xx} = \boldsymbol{0}.$$

The recent developments in computer algebra packages and improvements in CPU performances have further increased the number of m-component integrable systems with $U(m)$- or $O(m)$-invariance as well as $(m+1)$-component systems with $O(m)$-invariance to a considerable extent [9–11]. However, in contrast to the $U(m)$-invariant and $O(m)$-invariant systems, little research has been conducted on integrable systems having invariance with respect to the symplectic group $Sp(m)$; this refers to the group of $2m \times 2m$ real/complex symplectic matrices, $Sp(m,\mathbb{R})$ or $Sp(m,\mathbb{C})$, in accordance with the attribute of the dependent variables. To the best of the author's knowledge, the *only* example of a $2m$-component vector nonlinear PDE with $Sp(m)$ invariance is the system of coupled derivative mKdV equations studied using the bilinear method by Iwao and Hirota [12],

$$\frac{\partial u_i}{\partial t} + \frac{\partial^3 u_i}{\partial x^3} + 3\left[\sum_{j=1}^{m}\Big(\frac{\partial u_{2j-1}}{\partial x}u_{2j} - u_{2j-1}\frac{\partial u_{2j}}{\partial x}\Big)\right]\frac{\partial u_i}{\partial x} = 0, \quad i = 1, 2, \dots, 2m, \tag{1.4}$$

as well as its higher symmetries. It should be noted that system (1.4) is a natural multi-component generalization of the two-component system [(1.4) with $m = 1$] derived within the framework of the Sato theory by Loris and Willox [13, 14]. Here, the $Sp(m)$ invariance refers to the fact that system (1.4) is form-invariant under the following linear transformation: $(u_1, u_2, \ldots, u_{2m}) \mapsto (u_1, u_2, \ldots, u_{2m}) S^T$, where S is an (x, t)-independent element of the symplectic group $Sp(m)$ with a proper ordering of the base vectors, and the superscript ^{T} denotes the transposition.

The main objective of this paper is to expand the class of $Sp(m)$-invariant integrable systems and to characterize some of their interesting properties within the framework of the inverse scattering method. To achieve this goal, it must be first noted that system (1.4) is homogeneous with respect to the following weighting scheme: $w(\partial_x) = 1$, $w(\partial_t) = 3$, and $w(u_i) = 1/2$; this weighting scheme is the same as that for the third-order symmetries of vector derivative NLS (DNLS)-type systems. Motivated by this observation, we start with more general systems, that is, the third-order symmetries of integrable matrix generalizations of the DNLS-type equations. Then, we propose a *new* type of reduction for these third-order integrable matrix PDEs, at least in its explicit form, wherein one matrix variable is related to the transposition of the other multiplied by an antisymmetric constant matrix [15]. Considering the special case wherein the matrix variables are restricted to the form of column/row vectors, we obtain two $Sp(m)$-invariant systems; one coincides with system (1.4), while the other one,

$$\frac{\partial u_i}{\partial t} + \frac{\partial^3 u_i}{\partial x^3} + 3\frac{\partial}{\partial x}\left[\sum_{j=1}^{m}\Big(\frac{\partial u_{2j-1}}{\partial x}u_{2j} - u_{2j-1}\frac{\partial u_{2j}}{\partial x}\Big)u_i\right] = 0, \quad i = 1, 2, \ldots, 2m, \tag{1.5}$$

appears to be a new integrable system [15]. Note that the location of $\partial/\partial x$ in (1.5) is different from that in (1.4).

Once we have identified systems (1.4) and (1.5) as the reductions of the matrix DNLS-type systems, it is not difficult to further extend the class of $Sp(m)$-invariant integrable systems. First, we consider the integrable matrix generalizations [16, 17] of massive Thirring-type models [18–22], which are the first negative flows of the matrix DNLS hierarchies. Then, via the same type of reduction, we can directly obtain a new integrable variant of the massive Thirring model that is a hyperbolic system with $Sp(m)$-invariance,

$$\frac{\partial^2 v_i}{\partial \tau \partial x} + v_i - \left[\sum_{j=1}^{m}\Big(\frac{\partial v_{2j-1}}{\partial x}v_{2j} - v_{2j-1}\frac{\partial v_{2j}}{\partial x}\Big)\right]v_i = 0, \quad i = 1, 2, \ldots, 2m, \tag{1.6}$$

as well as an equivalent system up to the interchange of x and τ. Second, we consider the integrable space discretizations (semi-discretizations, for short) of the matrix DNLS hierarchies, including the massive Thirring-type models, proposed in ref. 23. Though not all of the semi-discrete matrix DNLS-type systems in ref. 23 are useful for our purpose, we find that the semi-discrete Kaup–Newell hierarchy [cf. (3.1) and (5.2) in ref. 23] allows proper reductions to yield integrable semi-discretizations of the above $Sp(m)$-invariant systems. Note that the discrete analogue of the reduction used to obtain a continuous $Sp(m)$-invariant system may not be uniquely determined. In fact, in addition to a single integrable semi-discretization of system (1.4), we obtain *two* integrable semi-discretizations for each of the $Sp(m)$-invariant systems (1.5) and (1.6). This result partly

illustrates the wide applicability of the semi-discrete Kaup–Newell hierarchy provided in ref. 23 to the theory of integrable discretizations.

Hereafter, we will not restrict ourselves to the *canonical* representation of the $Sp(m)$-invariant systems, but will present them in a slightly generalized (but still being integrable) form. Specifically, we replace terms such as $\sum_{j=1}^{m}\bigl(\frac{\partial u_{2j-1}}{\partial x}u_{2j}-u_{2j-1}\frac{\partial u_{2j}}{\partial x}\bigr)$ by those such as $\sum_{1\le j<k\le M}C_{jk}\bigl(\frac{\partial u_j}{\partial x}u_k-u_j\frac{\partial u_k}{\partial x}\bigr)$, where the integer M may or may not be even, and C_{jk} $(j<k)$ are arbitrary coupling constants (cf. ref. 12). It can be recalled that according to the classical theory of matrices (see, *e.g.*, refs. 24, 25), any antisymmetric matrix $C:=(C_{jk})_{j,k=1,\dots,M}$ can be transformed to the block diagonal form

$$P^TCP=\begin{pmatrix}\overbrace{J\quad\quad\quad\quad}^{m} & & \\ & \ddots & & \\ & & J & \\ & & & O\end{pmatrix},\quad J=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

with an invertible matrix P. Here, $2m\,(\le M)$ is equal to the rank of C, where m denotes the number of copies of J. This guarantees that through an invertible linear change of the dependent variables, the generalized system involving C_{jk} can always be converted into an $Sp(m)$-invariant (sub)system in the canonical form with linear equation(s) coupled to it, if any (the case of $2m<M$). Therefore, in this paper, we also use the term "symplectic invariance" for such generalized systems.

Using an approach based on the inverse scattering method, we can construct multi-soliton solutions of the matrix DNLS hierarchies under appropriate boundary conditions in both the continuous and discrete cases [26]. Thus, the soliton solutions for the $Sp(m)$-invariant systems contained in these hierarchies can be obtained by properly specializing the soliton parameters involved in the solutions for the latter. Moreover, as one of the main advantages of the inverse scattering method, the *most general* soliton solutions are obtained under the specified boundary conditions. However, the computations and discussions required to arrive at simple explicit formulas for the multi-soliton solutions are rather extensive and involved; hence, we do not present them here. We will defer the detailed derivation and investigation of the multi-soliton solutions to a subsequent publication [26]. In this paper, we assume decaying boundary conditions at spatial infinity and start with the linear integral/summation equations associated with the continuous/discrete matrix DNLS hierarchies, omitting their derivation via the inverse scattering method. These linear integral/summation equations, referred to as the Gel'fand–Levitan–Marchenko type, provide an *exact linearization* [27] of each nonlinear system under study in that they provide a relation between the solutions of the nonlinear system and those of the corresponding linear system. Considering a special case with a proper reduction, we solve the Gel'fand–Levitan–Marchenko equations to obtain the one-soliton solutions of the $Sp(m)$-invariant systems under the decaying boundary conditions. It should be emphasized that the soliton solutions of (1.4) obtained in this manner are indeed *more general* than the previously known solutions [12, 13]. Although the accuracy of these one-soliton solutions can easily be verified by direct substitutions, unlike the case of the NLS equation, it is not easy to obtain such solutions directly without resorting to the inverse scattering

method or other sophisticated methods in soliton theory. Indeed, any *naive* ansatz for the travelling wave solutions, *e.g.*, a complex plane wave modulated by a real envelope moving with a constant velocity, is most likely to fall into a trivial subclass of the most general one-soliton solutions, such as "the one-soliton solution" proposed by Loris and Willox [13].

This paper is organized as follows. In section 2, we demonstrate that the continuous systems (1.4), (1.5), and (1.6) in a slightly generalized form, as mentioned above, are obtained through a new type of reduction of the matrix DNLS hierarchies. We also present their bright one-soliton solutions. In section 3, we propose the integrable semi-discretizations of these continuous systems and present the one-soliton solutions for the most interesting semi-discrete systems. We also derive a new integrable semi-discretization of the Manakov model (1.1) from one of these semi-discrete systems. The last section, section 4, is devoted to concluding remarks, wherein we state that the reduction considered in this paper is not restricted to the DNLS-type systems, but is also applicable to a matrix generalization of the Yajima–Oikawa hierarchy.

2 Reductions of continuous matrix DNLS hierarchies

In this section, we propose the $Sp(m)$-invariant integrable systems via reductions of the continuous matrix derivative NLS hierarchies. We also present the Lax pairs, associated linear integral equations, and one-soliton solutions for the obtained systems.

2.1 Coupled derivative mKdV equations of Chen–Lee–Liu type

2.1.1 Lax pair for the second flow of the matrix Chen–Lee–Liu hierarchy

A derivative nonlinear Schrödinger (DNLS) equation $\mathrm{i}q_{t_2} + q_{xx} \pm \mathrm{i}|q|^2 q_x = 0$, which is often referred to as the Chen–Lee–Liu equation [28], permits an integrable matrix generalization [16, 17, 29, 30]

$$\begin{cases} \mathrm{i}q_{t_2} + q_{xx} - \mathrm{i}qrq_x = O, \\ \mathrm{i}r_{t_2} - r_{xx} - \mathrm{i}r_x qr = O. \end{cases} \tag{2.1}$$

Here, q and r are $l_1 \times l_2$ and $l_2 \times l_1$ matrices, respectively. Note that O on the right-hand side of the equations implies that the dependent variables can take their values in matrices. The lower indices of t are used to distinguish between different (and commutative) time evolutions; however, in the following text, we often omit these indices for brevity. In this paper, we are more concerned with the next higher flow in the matrix Chen–Lee–Liu hierarchy that commutes with the first nontrivial flow (2.1). It is written as (up to a scaling of the time variable) [16, 29, 30]

$$\begin{cases} q_{t_3} + q_{xxx} - \mathrm{i}\dfrac{3}{2}(q_x r q_x + qrq_{xx}) - \dfrac{3}{4}qrqrq_x = O, \\[2mm] r_{t_3} + r_{xxx} + \mathrm{i}\dfrac{3}{2}(r_x q r_x + r_{xx}qr) - \dfrac{3}{4}r_x qrqr = O. \end{cases} \tag{2.2}$$

The Lax pair for (2.2) is given by

$$U = \mathrm{i}\zeta^2 \begin{bmatrix} -I_1 & \\ & I_2 \end{bmatrix} + \zeta \begin{bmatrix} & q \\ r & \end{bmatrix} + \mathrm{i} \begin{bmatrix} O & \\ & \frac{1}{2} rq \end{bmatrix}, \tag{2.3a}$$

$$\begin{aligned} V = {} & \mathrm{i}\zeta^6 \begin{bmatrix} -4I_1 & \\ & 4I_2 \end{bmatrix} + \zeta^5 \begin{bmatrix} & 4q \\ 4r & \end{bmatrix} + \mathrm{i}\zeta^4 \begin{bmatrix} -2qr & \\ & 2rq \end{bmatrix} + \zeta^3 \begin{bmatrix} & 2\mathrm{i}q_x + qrq \\ -2\mathrm{i}r_x + rqr & \end{bmatrix} \\ & + \mathrm{i}\zeta^2 \begin{bmatrix} -\mathrm{i}(q_x r - q r_x) - \frac{1}{2}(qr)^2 & \\ & \mathrm{i}(r q_x - r_x q) + \frac{1}{2}(rq)^2 \end{bmatrix} \\ & + \zeta \begin{bmatrix} & -q_{xx} + \frac{\mathrm{i}}{2}(q_x r q - q r_x q + 2 q r q_x) + \frac{1}{4}(qr)^2 q \\ -r_{xx} - \frac{\mathrm{i}}{2}(2 r_x q r - r q_x r + r q r_x) + \frac{1}{4}(rq)^2 r & \end{bmatrix} \\ & + \mathrm{i} \begin{bmatrix} O & \\ & -\frac{1}{2}(r_{xx} q - r_x q_x + r q_{xx}) - \frac{\mathrm{i}}{4}(2 r_x q r q - r q_x r q + r q r_x q - 2 r q r q_x) + \frac{1}{8}(rq)^3 \end{bmatrix}. \end{aligned} \tag{2.3b}$$

Here, ζ is the spectral parameter independent of x and t; I_1 and I_2 are the $l_1 \times l_1$ and $l_2 \times l_2$ unit matrices, respectively. Let us substitute the Lax pair (2.3) in the zero-curvature condition [31, 32]

$$U_t - V_x + UV - VU = O, \tag{2.4}$$

which is the compatibility condition for the overdetermined system of linear PDEs,

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi. \tag{2.5}$$

Subsequently, equating the terms with the same powers of ζ to zero, we obtain the third-order matrix Chen–Lee–Liu system (2.2) without any contradiction or additional constraint.

2.1.2 Reduction

Both the first flow (2.1) and the second flow (2.2) of the matrix Chen–Lee–Liu hierarchy permit the reduction of the Hermitian conjugation [33] $r = A_1 q^\dagger A_2$, where A_1 and A_2 are constant Hermitian matrices: $A_i^\dagger = A_i$ and $A_{i,t} = A_{i,x} = O$. Moreover, unlike the first flow (2.1), the second flow (2.2) allows an interesting reduction such that r is identically equal to Cq^T, where C is an antisymmetric constant matrix: $C^T = -C$, $C_t = C_x = O$. In fact, the reduction $r = q^T C$ (or, more generally, $r = Bq^T C$, $B^T = B$, $C^T = -C$) can also be considered for (2.2), but we do not exploit this reduction in order to maintain a natural and easy-to-read flow of the paper. For the reduced system to assume a concise form without the imaginary unit and fractions, we consider the following vector reduction ($l_1 = 1$, $l_2 = M$):

$$q = (u_1, \ldots, u_M), \quad r = 2\mathrm{i}Cq^T = 2\mathrm{i} \begin{pmatrix} \sum_{k=1}^{M} C_{1k} u_k \\ \vdots \\ \sum_{k=1}^{M} C_{Mk} u_k \end{pmatrix}, \tag{2.6}$$

which also implies the simple relation $qr = 0$. System (2.2) is then reduced to a system of coupled derivative mKdV equations [12]:

$$\frac{\partial u_i}{\partial t} + \frac{\partial^3 u_i}{\partial x^3} + 3\left[\sum_{1\le j<k\le M} C_{jk}\left(\frac{\partial u_j}{\partial x}u_k - u_j\frac{\partial u_k}{\partial x}\right)\right]\frac{\partial u_i}{\partial x} = 0, \quad i = 1, 2, \ldots, M. \tag{2.7}$$

Using the vector notation, (2.7) can also be written as

$$\boldsymbol{u}_t + \boldsymbol{u}_{xxx} + 3\langle \boldsymbol{u}_x C, \boldsymbol{u}\rangle \boldsymbol{u}_x = \boldsymbol{0}, \quad C^T = -C.$$

Here, $\boldsymbol{u} = (u_1, u_2, \ldots, u_M)$ and $\langle \boldsymbol{u}_x C, \boldsymbol{u}\rangle = \boldsymbol{u}_x C \boldsymbol{u}^T$. The Lax pair for this reduced system is obtained by substituting (2.6) into q and r in (2.3); using a simple gauge transformation, the Lax pair can be rewritten in the form

$$U = \begin{bmatrix} -\lambda & \boldsymbol{u} \\ \lambda C\boldsymbol{u}^T & -C\boldsymbol{u}^T\boldsymbol{u} \end{bmatrix}, \tag{2.8a}$$

$$V = \left[\begin{array}{c|c} \lambda^3 + 2\lambda\langle \boldsymbol{u}_x C, \boldsymbol{u}\rangle & -\lambda^2\boldsymbol{u} + \lambda\boldsymbol{u}_x - \boldsymbol{u}_{xx} - 2\langle \boldsymbol{u}_x C, \boldsymbol{u}\rangle \boldsymbol{u} \\ \hline \begin{array}{l} -\lambda^3 C\boldsymbol{u}^T - \lambda^2 C\boldsymbol{u}_x^T \\ \quad - \lambda C\boldsymbol{u}_{xx}^T - 2\lambda\langle \boldsymbol{u}_x C, \boldsymbol{u}\rangle C\boldsymbol{u}^T \end{array} & \begin{array}{l} \lambda^2 C\boldsymbol{u}^T\boldsymbol{u} + \lambda C(\boldsymbol{u}_x^T\boldsymbol{u} - \boldsymbol{u}^T\boldsymbol{u}_x) \\ \quad + C(\boldsymbol{u}_{xx}^T\boldsymbol{u} - \boldsymbol{u}_x^T\boldsymbol{u}_x + \boldsymbol{u}^T\boldsymbol{u}_{xx}) + 2\langle \boldsymbol{u}_x C, \boldsymbol{u}\rangle C\boldsymbol{u}^T\boldsymbol{u} \end{array} \end{array}\right], \tag{2.8b}$$

where $\lambda := 2\mathrm{i}\zeta^2$ is the “new” spectral parameter. In the simplest nontrivial case of $M = 2$, setting

$$C_{12} = \mathrm{i}, \quad u_1 = \psi, \quad u_2 = \psi^*,$$

we obtain the following single equation [13, 14]:

$$\psi_t + \psi_{xxx} + 3\mathrm{i}(\psi_x\psi^* - \psi\psi_x^*)\psi_x = 0. \tag{2.9}$$

Here, the asterisk denotes the complex conjugate. Using a simple point transformation (cf. refs. 34, 35), we can convert (2.9) into an NLS-type equation perturbed by higher order terms.

2.1.3 Relation to the second flow of the matrix NLS hierarchy

The matrix DNLS hierarchies, including the matrix Chen–Lee–Liu hierarchy, can be considered embedded in a generalization of the matrix NLS hierarchy [23]. In the case of reduction (2.6), the particular relation $qr = 0$ simplifies the embedding formulas to a considerable extent. As a result, (2.7) can be derived from the second nontrivial flow of the matrix NLS hierarchy, i.e., a matrix analogue of the (non-reduced) complex mKdV equation [36, 37]

$$\begin{cases} Q_t + Q_{xxx} - 3Q_xRQ - 3QRQ_x = O, \\ R_t + R_{xxx} - 3R_xQR - 3RQR_x = O, \end{cases} \tag{2.10}$$

through a simple, but not ultralocal, reduction. Indeed, if we set

$$Q = (u_1, \ldots, u_M), \quad R = CQ_x^T = \begin{pmatrix} \sum_{k=1}^M C_{1k}u_{k,x} \\ \vdots \\ \sum_{k=1}^M C_{Mk}u_{k,x} \end{pmatrix}, \tag{2.11}$$

in terms of an antisymmetric constant matrix C, we obtain the following relations:

$$Q_x R = 0, \quad QR = -\sum_{1\le j<k\le M} C_{jk}(u_{j,x}u_k - u_j u_{k,x}), \quad QR_x = (QR)_x.$$

Thus, the two matrix equations (2.10) simply collapse to form a single vector equation (2.7).

There exist a few advantages in regarding (2.7) as a reduced form of (2.10). First, an infinite set of conservation laws for (2.10) can be constructed systematically and rather easily using a recursive formula based on the Lax pair [38]. Thus, we can obtain the conservation laws for (2.7) from those for (2.10) through the reduction (2.11). The conserved densities of the first two ranks obtained in this manner are given by

$$u_{j,x}u_k \ \ (j\neq k), \qquad \sum_{1\le j<k\le M} C_{jk}(u_{j,xx}u_{k,x} - u_{j,x}u_{k,xx}) - \left[\sum_{1\le j<k\le M} C_{jk}(u_{j,x}u_k - u_j u_{k,x})\right]^2,$$

which are derived from the conserved densities RQ and $\mathrm{tr}\,(Q_x R_x + QRQR)$ of (2.10), respectively. Second, considering a similar reduction for a space-discrete analogue of (2.10) [23], we can obtain an integrable semi-discretization of the continuous system (2.7). We employ this approach in subsection 3.1 since none of the (semi-)discrete analogues of the third-order matrix Chen–Lee–Liu system (2.2), which are integrable and permit a reduction like (2.6), are known.

2.1.4 Solution formulas

An approach based on the inverse scattering method [26] enables the derivation of the solutions of the matrix Chen–Lee–Liu hierarchy, that is, (2.1) and (2.2) as well as the higher flows, through a set of formulas (cf. ref. 39 for the case of scalar variables)

$$q = K(x,x), \tag{2.12a}$$

$$r = \bar{K}(x,x), \tag{2.12b}$$

$$K(x,y) = \bar{F}(y) + \frac{\mathrm{i}}{2}\int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2\, K(x,s_1)F(s_1+s_2-x)\frac{\partial \bar{F}(s_2+y-x)}{\partial s_2}, \quad y\ge x, \tag{2.12c}$$

$$\bar{K}(x,y) = F(y) - \frac{\mathrm{i}}{2}\int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2\, \frac{\partial \bar{K}(x,s_1)}{\partial s_1}\bar{F}(s_1+s_2-x)F(s_2+y-x), \quad y\ge x. \tag{2.12d}$$

Here and hereafter, the bar does *not* denote the complex/Hermitian conjugate in general. The time dependence of the functions is suppressed in formulas (2.12). The functions $\bar{F}(x)$ and $F(x)$ satisfy the corresponding *linear* uncoupled system of matrix PDEs, *e.g.*,

$$\mathrm{i}\frac{\partial \bar{F}}{\partial t_2} + \frac{\partial^2 \bar{F}}{\partial x^2} = O, \quad \mathrm{i}\frac{\partial F}{\partial t_2} - \frac{\partial^2 F}{\partial x^2} = O \tag{2.13}$$

for the second-order matrix Chen–Lee–Liu system (2.1) and

$$\frac{\partial \bar{F}}{\partial t_3} + \frac{\partial^3 \bar{F}}{\partial x^3} = O, \quad \frac{\partial F}{\partial t_3} + \frac{\partial^3 F}{\partial x^3} = O \tag{2.14}$$

for the third-order matrix Chen–Lee–Liu flow (2.2), and decay rapidly as $x \to +\infty$. Note that formulas (2.12) involve only equal-time quantities and depend on the time variables through the time evolution of $\bar{F}$ and F.

The reduction (2.6) is achieved at the level of the solution formulas by setting

$$\bar{F}(x,t) = (f_1, f_2, \dots, f_M)(x,t) =: \boldsymbol{f}(x,t), \quad F(x,t) = 2\mathrm{i}C\bar{F}(x,t)^T.$$

With this reduction, the set of formulas (2.12) is reduced to a compact form. Thus, the solutions to the coupled derivative mKdV equations (2.7), decaying as $x \to +\infty$, can be constructed from those of the linear vector PDE $\boldsymbol{f}_t + \boldsymbol{f}_{xxx} = \boldsymbol{0}$ through the formula

$$\boldsymbol{u}(x,t) = \boldsymbol{k}(x,x;t), \tag{2.15a}$$

$$\boldsymbol{k}(x,y) = \boldsymbol{f}(y) - \int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2\, \boldsymbol{k}(x,s_1) C \boldsymbol{f}(s_1+s_2-x)^T \frac{\partial \boldsymbol{f}(s_2+y-x)}{\partial s_2}, \quad y \geq x. \tag{2.15b}$$

Here, $\boldsymbol{u} = (u_1, u_2, \dots, u_M)$ and $\boldsymbol{k}(x,y)$ are M-component row vectors. Note that for a given $\boldsymbol{f}$, the integral equation (2.15b), referred to as the Gel'fand–Levitan–Marchenko type, is linear for unknown $\boldsymbol{k}$. We assume that both $\boldsymbol{u}$ and the eigenfunctions bound in the potential $\boldsymbol{u}$ (cf. (2.5)) should also decay as $x \to -\infty$. Then, we can verify that the number of distinct exponential functions comprising $\boldsymbol{f}(x,t)$ has to be even. Substituting

$$\boldsymbol{f}(x,t) = \boldsymbol{a}_1 \mathrm{e}^{\mathrm{i}\lambda_1 x + \mathrm{i}\lambda_1^3 t} + \boldsymbol{a}_2 \mathrm{e}^{\mathrm{i}\lambda_2 x + \mathrm{i}\lambda_2^3 t}, \quad \mathrm{Im}\,\lambda_j > 0 \ (j=1,2), \quad \lambda_1 \neq \lambda_2, \quad \langle \boldsymbol{a}_1 C, \boldsymbol{a}_2 \rangle := \boldsymbol{a}_1 C \boldsymbol{a}_2^T \neq 0,$$
$$\boldsymbol{k}(x,y;t) = \boldsymbol{k}_1(x,t)\mathrm{e}^{\mathrm{i}\lambda_1 y + \mathrm{i}\lambda_1^3 t} + \boldsymbol{k}_2(x,t)\mathrm{e}^{\mathrm{i}\lambda_2 y + \mathrm{i}\lambda_2^3 t}$$

into (2.15) and solving it with respect to $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, we obtain the "unrefined" one-soliton solution of system (2.7),

$$\begin{aligned} \boldsymbol{u}(x,t) &= \boldsymbol{k}_1(x,t)\mathrm{e}^{\mathrm{i}\lambda_1 x + \mathrm{i}\lambda_1^3 t} + \boldsymbol{k}_2(x,t)\mathrm{e}^{\mathrm{i}\lambda_2 x + \mathrm{i}\lambda_2^3 t} \\ &= \frac{\boldsymbol{a}_1 \mathrm{e}^{\mathrm{i}\lambda_1 x + \mathrm{i}\lambda_1^3 t} + \boldsymbol{a}_2 \mathrm{e}^{\mathrm{i}\lambda_2 x + \mathrm{i}\lambda_2^3 t}}{1 - \frac{\mathrm{i}(\lambda_1-\lambda_2)}{2(\lambda_1+\lambda_2)^2} \langle \boldsymbol{a}_1 C, \boldsymbol{a}_2 \rangle \mathrm{e}^{\mathrm{i}(\lambda_1+\lambda_2)x + \mathrm{i}(\lambda_1^3+\lambda_2^3)t}}. \end{aligned} \tag{2.16}$$

Note that the denominator in the above expression may become zero for certain values of x and t. By introducing a new parametrization,

$$-\frac{\mathrm{i}(\lambda_1-\lambda_2)}{2(\lambda_1+\lambda_2)^2} \langle \boldsymbol{a}_1 C, \boldsymbol{a}_2 \rangle =: \mathrm{e}^{-2\delta} \ \ (\delta \in \mathbb{C}), \quad \boldsymbol{a}_1 =: 2\mathrm{e}^{-\delta}\boldsymbol{b}_1, \quad \boldsymbol{a}_2 =: 2\mathrm{e}^{-\delta}\boldsymbol{b}_2,$$

(2.16) can be rewritten as

$$\boldsymbol{u}(x,t) = \frac{\boldsymbol{b}_1 \mathrm{e}^{\frac{\mathrm{i}}{2}(\lambda_1-\lambda_2)x + \frac{\mathrm{i}}{2}(\lambda_1^3-\lambda_2^3)t} + \boldsymbol{b}_2 \mathrm{e}^{-\frac{\mathrm{i}}{2}(\lambda_1-\lambda_2)x - \frac{\mathrm{i}}{2}(\lambda_1^3-\lambda_2^3)t}}{\cosh\left[\frac{\mathrm{i}}{2}(\lambda_1+\lambda_2)x + \frac{\mathrm{i}}{2}(\lambda_1^3+\lambda_2^3)t - \delta\right]}, \tag{2.17}$$

with the condition $-2\mathrm{i}(\lambda_1-\lambda_2)\langle \boldsymbol{b}_1 C, \boldsymbol{b}_2 \rangle = (\lambda_1+\lambda_2)^2$. The "one-soliton" solution (2.17) resembles the soliton solution of the vector mKdV equation (1.2) [38],

$$\boldsymbol{q}(x,t) = \frac{\boldsymbol{c}_1 \mathrm{e}^{\frac{\mathrm{i}}{2}(\lambda_1-\lambda_2)x + \frac{\mathrm{i}}{2}(\lambda_1^3-\lambda_2^3)t} + \boldsymbol{c}_2 \mathrm{e}^{-\frac{\mathrm{i}}{2}(\lambda_1-\lambda_2)x - \frac{\mathrm{i}}{2}(\lambda_1^3-\lambda_2^3)t}}{\cosh\left[\frac{\mathrm{i}}{2}(\lambda_1+\lambda_2)x + \frac{\mathrm{i}}{2}(\lambda_1^3+\lambda_2^3)t - \delta\right]},$$

under the conditions $\langle \boldsymbol{c}_1, \boldsymbol{c}_1\rangle = \langle \boldsymbol{c}_2, \boldsymbol{c}_2\rangle = 0$ and $-8\langle \boldsymbol{c}_1, \boldsymbol{c}_2\rangle = (\lambda_1+\lambda_2)^2$. Moreover, if we impose the "reality conditions" $\lambda_2 = -\lambda_1^*$ and $\mathrm{e}^{2\delta} \notin \mathbb{R}_{<0}$, (2.17) provides the bright one-soliton solution of system (2.7) that behaves regularly for real x and t. With the parametrization $\lambda_1 = \xi_1 + \mathrm{i}\eta_1$, $\lambda_2 = -\lambda_1^* = -\xi_1 + \mathrm{i}\eta_1$ ($\xi_1 \neq 0$, $\eta_1 > 0$), it reads as

$$\boldsymbol{u}(x,t) = \frac{\boldsymbol{b}_1 \mathrm{e}^{\mathrm{i}\xi_1 x + \mathrm{i}\xi_1(\xi_1^2 - 3\eta_1^2)t} + \boldsymbol{b}_2 \mathrm{e}^{-\mathrm{i}\xi_1 x - \mathrm{i}\xi_1(\xi_1^2-3\eta_1^2)t}}{\cosh\left[\eta_1 x + \eta_1(3\xi_1^2 - \eta_1^2)t + \delta\right]}, \quad \mathrm{i}\xi_1\langle \boldsymbol{b}_1 C, \boldsymbol{b}_2\rangle = \eta_1^2, \quad \mathrm{Im}\,(2\delta) \not\equiv \pi \pmod{2\pi}.$$

Similarly, the general N-soliton solution before imposing the "reality conditions" is obtained by substituting the expressions

$$\boldsymbol{f}(x,t) = \sum_{l=1}^{2N} \boldsymbol{a}_l \mathrm{e}^{\mathrm{i}\lambda_l x + \mathrm{i}\lambda_l^3 t}, \quad \mathrm{Im}\,\lambda_l > 0, \quad \lambda_j \neq \lambda_k \text{ if } j \neq k,$$
$$\boldsymbol{k}(x,y;t) = \sum_{l=1}^{2N} \boldsymbol{k}_l(x,t) \mathrm{e}^{\mathrm{i}\lambda_l y + \mathrm{i}\lambda_l^3 t}$$

into (2.15) and solving the resulting linear algebraic system for $\boldsymbol{k}_l$ ($l = 1, 2, \ldots, 2N$) using a matrix inversion. In addition, conditions such as

$$\text{Pfaffian of } \left(\frac{\boldsymbol{a}_j C \boldsymbol{a}_k^T}{\lambda_j + \lambda_k}\right) \neq 0$$

have to be imposed for the solution to decay as $x \to -\infty$ and to behave properly as solitons. The details will be published elsewhere. It should be noted that the soliton solutions thus obtained involve more free parameters, and hence, are more general than the previously known solutions [12, 13].

2.2 Coupled derivative mKdV equations of Kaup–Newell type

2.2.1 Lax pair for the second flow of the matrix Kaup–Newell hierarchy

Another DNLS equation $\mathrm{i}q_{t_2} + q_{xx} \pm \mathrm{i}(|q|^2 q)_x = 0$, which is often referred to as the Kaup–Newell equation [40], permits an integrable matrix generalization [16, 17, 30, 41]

$$\begin{cases} \mathrm{i}q_{t_2} + q_{xx} - \mathrm{i}(qrq)_x = O, \\ \mathrm{i}r_{t_2} - r_{xx} - \mathrm{i}(rqr)_x = O. \end{cases} \tag{2.18}$$

Here, q and r are $l_1 \times l_2$ and $l_2 \times l_1$ matrices, respectively. In this paper, we are more concerned with the next higher flow in the matrix Kaup–Newell hierarchy that commutes with the first nontrivial flow (2.18). It is written as (up to a scaling of the time variable) [16]

$$\begin{cases} q_{t_3} + q_{xxx} - \mathrm{i}\dfrac{3}{2}(q_x r q + q r q_x)_x - \dfrac{3}{2}(qrqrq)_x = O, \\ r_{t_3} + r_{xxx} + \mathrm{i}\dfrac{3}{2}(r_x q r + r q r_x)_x - \dfrac{3}{2}(rqrqr)_x = O. \end{cases} \tag{2.19}$$

Note that the lower indices of t are omitted in the following. The Lax pair for (2.19) is given by

$$U = \mathrm{i}\zeta^2 \begin{bmatrix} -I_1 & \\ & I_2 \end{bmatrix} + \zeta \begin{bmatrix} & q \\ r & \end{bmatrix}, \tag{2.20a}$$

$$\begin{aligned} V = {} & \mathrm{i}\zeta^6 \begin{bmatrix} -4I_1 & \\ & 4I_2 \end{bmatrix} + \zeta^5 \begin{bmatrix} & 4q \\ 4r & \end{bmatrix} + \mathrm{i}\zeta^4 \begin{bmatrix} -2qr & \\ & 2rq \end{bmatrix} \\ & + \zeta^3 \begin{bmatrix} & 2\mathrm{i}q_x + 2qrq \\ -2\mathrm{i}r_x + 2rqr & \end{bmatrix} \\ & + \mathrm{i}\zeta^2 \begin{bmatrix} -\mathrm{i}(q_x r - q r_x) - \frac{3}{2}(qr)^2 & \\ & \mathrm{i}(r q_x - r_x q) + \frac{3}{2}(rq)^2 \end{bmatrix} \\ & + \zeta \begin{bmatrix} & -q_{xx} + \mathrm{i}\frac{3}{2}(q_x r q + q r q_x) + \frac{3}{2}(qr)^2 q \\ -r_{xx} - \mathrm{i}\frac{3}{2}(r_x q r + r q r_x) + \frac{3}{2}(rq)^2 r & \end{bmatrix}. \end{aligned} \tag{2.20b}$$

Substituting the Lax pair (2.20) in the zero-curvature condition (2.4), we obtain the third-order matrix Kaup–Newell system (2.19) without any contradiction or additional constraint.

2.2.2 Reduction

Both the first flow (2.18) and the second flow (2.19) of the matrix Kaup–Newell hierarchy permit the reduction of the Hermitian conjugation [33] $r = A_1 q^\dagger A_2$, where A_1 and A_2 are constant Hermitian matrices: $A_i^\dagger = A_i$, $A_{i,t} = A_{i,x} = O$. Moreover, unlike the first flow (2.18), the second flow (2.19) allows an interesting reduction such that r is identically equal to Cq^T, where C is an antisymmetric constant matrix: $C^T = -C$, $C_t = C_x = O$. For the reduced system to assume a concise form without the imaginary unit and fractions, the following vector reduction ($l_1 = 1$, $l_2 = M$) is considered:

$$q = (u_1, \dots, u_M), \qquad r = 2\mathrm{i}Cq^T = 2\mathrm{i} \begin{pmatrix} \sum_{k=1}^{M} C_{1k} u_k \\ \vdots \\ \sum_{k=1}^{M} C_{Mk} u_k \end{pmatrix}, \tag{2.21}$$

which also implies the simple relation $qr = 0$. System (2.19) is then reduced to another system of coupled derivative mKdV equations:

$$\frac{\partial u_i}{\partial t} + \frac{\partial^3 u_i}{\partial x^3} + 3\frac{\partial}{\partial x}\left[\sum_{1 \le j < k \le M} C_{jk} \left(\frac{\partial u_j}{\partial x} u_k - u_j \frac{\partial u_k}{\partial x} \right) u_i \right] = 0, \quad i = 1, 2, \dots, M. \tag{2.22}$$

To the best of the author's knowledge, this system was reported for the first time in [15]. In contrast to (2.7), the partial differentiation with respect to x acts on the entire nonlinear

term in (2.22). The conserved densities of the first three ranks are given by

$$u_i \ \ (i=1,2,\dots,M),$$

$$\sum_{1\le j<k\le M} C_{jk}(u_{j,x}u_k - u_j u_{k,x}),$$

$$\sum_{1\le j<k\le M} C_{jk}(u_{j,xx}u_{k,x} - u_{j,x}u_{k,xx}) - 2\left[\sum_{1\le j<k\le M} C_{jk}(u_{j,x}u_k - u_j u_{k,x})\right]^2.$$

Using the vector notation, (2.22) can also be written as

$$\boldsymbol{u}_t + \boldsymbol{u}_{xxx} + 3\big(\langle \boldsymbol{u}_x C, \boldsymbol{u}\rangle \boldsymbol{u}\big)_x = \boldsymbol{0}, \quad C^T = -C.$$

The Lax pair for this reduced system is obtained by substituting (2.21) into q and r in (2.20); using a simple gauge transformation, it can be rewritten in the form

$$U = \left[\begin{array}{cc} -\lambda & \boldsymbol{u} \\ \lambda C\boldsymbol{u}^T & O \end{array}\right], \tag{2.23a}$$

$$V = \left[\begin{array}{c|c} \lambda^3 + 2\lambda\langle \boldsymbol{u}_x C, \boldsymbol{u}\rangle & -\lambda^2\boldsymbol{u} + \lambda\boldsymbol{u}_x - \boldsymbol{u}_{xx} - 3\langle \boldsymbol{u}_x C, \boldsymbol{u}\rangle\boldsymbol{u} \\ \hline \begin{array}{c} -\lambda^3 C\boldsymbol{u}^T - \lambda^2 C\boldsymbol{u}_x^T \\ -\lambda C\boldsymbol{u}_{xx}^T - 3\lambda\langle \boldsymbol{u}_x C, \boldsymbol{u}\rangle C\boldsymbol{u}^T \end{array} & \lambda^2 C\boldsymbol{u}^T\boldsymbol{u} + \lambda C(\boldsymbol{u}_x^T\boldsymbol{u} - \boldsymbol{u}^T\boldsymbol{u}_x) \end{array}\right], \tag{2.23b}$$

where $\boldsymbol{u}$ is a row vector and $\lambda := 2\mathrm{i}\zeta^2$. In the simplest nontrivial case of $M=2$, setting

$$C_{12} = \mathrm{i}, \quad u_1 = \psi, \quad u_2 = \psi^*,$$

the following single equation is obtained:

$$\psi_t + \psi_{xxx} + 3\mathrm{i}\big[(\psi_x\psi^* - \psi\psi_x^*)\psi\big]_x = 0. \tag{2.24}$$

Using a simple point transformation (cf. refs. 34, 35), (2.24) can be converted into an NLS-type equation perturbed by higher order terms.

2.2.3 Solution formulas

A set of formulas for the solutions of the matrix Kaup–Newell hierarchy, decaying as $x \to +\infty$, can be derived by exploiting its relationship [16, 17] with the matrix Chen–Lee–Liu hierarchy studied in subsection 2.1. It is written in the form of the product of

two matrices as follows [26] (cf. ref. 39 for the case of scalar variables):

$$q = K(x,x)\left[I + L(x,x)\right], \tag{2.25a}$$

$$r = \left[I + L(x,x)\right]^{-1} \bar{K}(x,x), \tag{2.25b}$$

$$K(x,y) = \bar{F}(y) + \frac{\mathrm{i}}{2}\int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2\, K(x,s_1)F(s_1+s_2-x)\frac{\partial \bar{F}(s_2+y-x)}{\partial s_2}, \quad y \geq x, \tag{2.25c}$$

$$\bar{K}(x,y) = F(y) - \frac{\mathrm{i}}{2}\int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2\, \frac{\partial \bar{K}(x,s_1)}{\partial s_1}\bar{F}(s_1+s_2-x)F(s_2+y-x), \quad y \geq x, \tag{2.25d}$$

$$\begin{aligned} L(x,y) = &-\frac{\mathrm{i}}{2}\int_x^\infty \mathrm{d}s\, F(s)\bar{F}(s+y-x) \\ &+ \frac{\mathrm{i}}{2}\int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2\, \frac{\partial L(x,s_1)}{\partial s_1}F(s_1+s_2-x)\bar{F}(s_2+y-x), \quad y \geq x. \end{aligned} \tag{2.25e}$$

The time dependence of the functions as well as the index of the unit matrix I to indicate its size is suppressed in the above formulas. Here, $\bar{F}(x)$ and $F(x)$ satisfy the corresponding *linear* uncoupled system of matrix PDEs, *e.g.*, (2.13) for the matrix Kaup–Newell system (2.18) and (2.14) for the third-order flow (2.19), and decay rapidly as $x \to +\infty$. This set of formulas is indeed useful and sufficient for constructing explicit solutions of the matrix Kaup–Newell hierarchy. However, its shortcoming is that one of the most important properties of the hierarchy has not been incorporated, that is, the hierarchy allows the introduction of the potential variables $q =: \hat{q}_x$ and $r =: \hat{r}_x$, and can be reformulated in terms of $\hat{q}$ and $\hat{r}$. Consequently, the elementary function solutions of the matrix Kaup–Newell hierarchy can be expressed as the partial x-derivatives of elementary functions; this fact has *passed unnoticed* in the existing literature (see refs. 16, 22 for some indirect results). To bridge this gap, we propose a new set of solution formulas that accurately reflects the feasibility of potentiation for the matrix Kaup–Newell hierarchy; it assumes the following form (x-derivatives of single quantities [16, 22]):

$$q = \frac{\partial \mathcal{K}(x,x)}{\partial x}, \tag{2.26a}$$

$$r = \frac{\partial \bar{\mathcal{K}}(x,x)}{\partial x}, \tag{2.26b}$$

$$\begin{aligned} \mathcal{K}(x,y) &= -\int_y^\infty \mathrm{d}s_1\, \bar{F}(s_1) + \frac{\mathrm{i}}{2}\int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2\, \frac{\partial \mathcal{K}(x,s_1)}{\partial s_1}F(s_1+s_2-x)\bar{F}(s_2+y-x) \\ &= \bar{G}(y) + \frac{\mathrm{i}}{2}\int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2\, \frac{\partial \mathcal{K}(x,s_1)}{\partial s_1}\frac{\partial G(s_1+s_2-x)}{\partial s_2}\frac{\partial \bar{G}(s_2+y-x)}{\partial y}, \quad y \geq x, \end{aligned} \tag{2.26c}$$

$$\begin{aligned} \bar{\mathcal{K}}(x,y) &= -\int_y^\infty \mathrm{d}s_1\, F(s_1) - \frac{\mathrm{i}}{2}\int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2\, \frac{\partial \bar{\mathcal{K}}(x,s_1)}{\partial s_1}\bar{F}(s_1+s_2-x)F(s_2+y-x) \\ &= G(y) - \frac{\mathrm{i}}{2}\int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2\, \frac{\partial \bar{\mathcal{K}}(x,s_1)}{\partial s_1}\frac{\partial \bar{G}(s_1+s_2-x)}{\partial s_2}\frac{\partial G(s_2+y-x)}{\partial y}, \quad y \geq x. \end{aligned} \tag{2.26d}$$

Here, the matrices $\bar{G}$ and G are the primitive functions of $\bar{F}$ and F, respectively, that also decay as $x \to +\infty$, that is, $\bar{G}(x) := -\int_x^\infty \bar{F}(y)\,\mathrm{d}y$ and $G(x) := -\int_x^\infty F(y)\,\mathrm{d}y$. Note that $\partial/\partial x$ in (2.26a) and (2.26b) denotes the partial differentiation with respect to x, while all the time variables are fixed. The two sets of formulas (2.25) and (2.26) are indeed equivalent, though it is a highly nontrivial task to verify it directly.

Similarly to the case of the matrix Chen–Lee–Liu hierarchy, the reduction (2.21) can be realized by setting

$$\bar{G}(x,t) = (g_1, g_2, \dots, g_M)(x,t) =: \boldsymbol{g}(x,t), \quad G(x,t) = 2\mathrm{i}C\bar{G}(x,t)^T. \tag{2.27}$$

Thus, the set of formulas (2.26) is reduced to a compact form. In particular, the solutions to the coupled derivative mKdV equations (2.22), decaying as $x \to +\infty$, can be constructed from those of the linear vector PDE $\boldsymbol{g}_t + \boldsymbol{g}_{xxx} = \boldsymbol{0}$ through the formula

$$\boldsymbol{u}(x,t) = \frac{\partial}{\partial x}\boldsymbol{k}(x,x;t), \tag{2.28a}$$

$$\boldsymbol{k}(x,y) = \boldsymbol{g}(y) - \int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2 \frac{\partial \boldsymbol{k}(x,s_1)}{\partial s_1} C \frac{\partial \boldsymbol{g}(s_1+s_2-x)^T}{\partial s_2} \frac{\partial \boldsymbol{g}(s_2+y-x)}{\partial y}, \quad y \geq x. \tag{2.28b}$$

Here, $\boldsymbol{u} = (u_1, u_2, \dots, u_M)$ and $\boldsymbol{k}(x,y)$ are M-component row vectors. Substituting the expressions

$$\boldsymbol{g}(x,t) = \boldsymbol{a}_1 \mathrm{e}^{\mathrm{i}\lambda_1 x + \mathrm{i}\lambda_1^3 t} + \boldsymbol{a}_2 \mathrm{e}^{\mathrm{i}\lambda_2 x + \mathrm{i}\lambda_2^3 t}, \quad \mathrm{Im}\,\lambda_j > 0 \ (j=1,2), \quad \lambda_1 \neq \lambda_2, \quad \langle \boldsymbol{a}_1 C, \boldsymbol{a}_2 \rangle := \boldsymbol{a}_1 C \boldsymbol{a}_2^T \neq 0,$$
$$\boldsymbol{k}(x,y;t) = \boldsymbol{k}_1(x,t)\mathrm{e}^{\mathrm{i}\lambda_1 y + \mathrm{i}\lambda_1^3 t} + \boldsymbol{k}_2(x,t)\mathrm{e}^{\mathrm{i}\lambda_2 y + \mathrm{i}\lambda_2^3 t}$$

into (2.28) and solving it with respect to $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, we obtain the "unrefined" one-soliton solution of system (2.22),

$$\begin{aligned}\boldsymbol{u}(x,t) &= \frac{\partial}{\partial x}\left[\boldsymbol{k}_1(x,t)\mathrm{e}^{\mathrm{i}\lambda_1 x + \mathrm{i}\lambda_1^3 t} + \boldsymbol{k}_2(x,t)\mathrm{e}^{\mathrm{i}\lambda_2 x + \mathrm{i}\lambda_2^3 t}\right] \\ &= \frac{\partial}{\partial x}\left[\frac{\boldsymbol{a}_1 \mathrm{e}^{\mathrm{i}\lambda_1 x + \mathrm{i}\lambda_1^3 t} + \boldsymbol{a}_2 \mathrm{e}^{\mathrm{i}\lambda_2 x + \mathrm{i}\lambda_2^3 t}}{1 + \frac{\mathrm{i}\lambda_1\lambda_2(\lambda_1-\lambda_2)}{2(\lambda_1+\lambda_2)^2}\langle \boldsymbol{a}_1 C, \boldsymbol{a}_2\rangle \mathrm{e}^{\mathrm{i}(\lambda_1+\lambda_2)x + \mathrm{i}(\lambda_1^3+\lambda_2^3)t}}\right].\end{aligned} \tag{2.29}$$

Note that the above solution also decays as $x \to -\infty$, but may have singularities for certain values of x and t. By introducing a new parametrization,

$$\frac{\mathrm{i}\lambda_1\lambda_2(\lambda_1-\lambda_2)}{2(\lambda_1+\lambda_2)^2}\langle \boldsymbol{a}_1 C, \boldsymbol{a}_2\rangle =: \mathrm{e}^{-2\delta} \ (\delta \in \mathbb{C}), \quad \boldsymbol{a}_1 =: 2\mathrm{e}^{-\delta}\boldsymbol{b}_1, \quad \boldsymbol{a}_2 =: 2\mathrm{e}^{-\delta}\boldsymbol{b}_2,$$

(2.29) can be rewritten as

$$\boldsymbol{u}(x,t) = \frac{\partial}{\partial x}\left\{\frac{\boldsymbol{b}_1 \mathrm{e}^{\frac{\mathrm{i}}{2}(\lambda_1-\lambda_2)x + \frac{\mathrm{i}}{2}(\lambda_1^3-\lambda_2^3)t} + \boldsymbol{b}_2 \mathrm{e}^{-\frac{\mathrm{i}}{2}(\lambda_1-\lambda_2)x - \frac{\mathrm{i}}{2}(\lambda_1^3-\lambda_2^3)t}}{\cosh\left[\frac{\mathrm{i}}{2}(\lambda_1+\lambda_2)x + \frac{\mathrm{i}}{2}(\lambda_1^3+\lambda_2^3)t - \delta\right]}\right\}, \tag{2.30}$$

with the condition $2\mathrm{i}\lambda_1\lambda_2(\lambda_1-\lambda_2)\langle \boldsymbol{b}_1 C, \boldsymbol{b}_2\rangle = (\lambda_1+\lambda_2)^2$. Moreover, if we impose the "reality conditions" $\lambda_2 = -\lambda_1^*$ and $\mathrm{e}^{2\delta} \notin \mathbb{R}_{<0}$, (2.30) provides the bright one-soliton solution of system (2.22) that behaves regularly for real x and t. With the parametrization

$\lambda_1 = \xi_1 + \mathrm{i}\eta_1$, $\lambda_2 = -\lambda_1^* = -\xi_1 + \mathrm{i}\eta_1$ ($\xi_1 \neq 0$, $\eta_1 > 0$), it reads as

$$
\begin{aligned}
\boldsymbol{u}(x,t) &= \frac{\partial}{\partial x}\left\{ \frac{\boldsymbol{b}_1 \mathrm{e}^{\mathrm{i}\xi_1 x + \mathrm{i}\xi_1(\xi_1^2 - 3\eta_1^2)t} + \boldsymbol{b}_2 \mathrm{e}^{-\mathrm{i}\xi_1 x - \mathrm{i}\xi_1(\xi_1^2-3\eta_1^2)t}}{\cosh\left[\eta_1 x + \eta_1(3\xi_1^2 - \eta_1^2)t + \delta\right]} \right\} \\
&= \frac{\mathrm{i}\sqrt{\xi_1^2+\eta_1^2}}{\cosh^2\left[\eta_1 x + \eta_1(3\xi_1^2-\eta_1^2)t + \delta\right]} \Big\{ \boldsymbol{b}_1 \cosh\left[\eta_1 x + \eta_1(3\xi_1^2 - \eta_1^2)t + \delta + \mathrm{i}\varphi\right] \mathrm{e}^{\mathrm{i}\xi_1 x + \mathrm{i}\xi_1(\xi_1^2-3\eta_1^2)t} \\
&\qquad - \boldsymbol{b}_2 \cosh\left[\eta_1 x + \eta_1(3\xi_1^2-\eta_1^2)t + \delta - \mathrm{i}\varphi\right] \mathrm{e}^{-\mathrm{i}\xi_1 x - \mathrm{i}\xi_1(\xi_1^2-3\eta_1^2)t} \Big\},
\end{aligned}
$$

with the conditions $\mathrm{i}\xi_1\langle \boldsymbol{b}_1 C, \boldsymbol{b}_2\rangle = \eta_1^2/(\xi_1^2+\eta_1^2)$, $\exp(\mathrm{i}\varphi) := (\xi_1 + \mathrm{i}\eta_1)/\sqrt{\xi_1^2+\eta_1^2}$, and $\mathrm{Im}(2\delta) \not\equiv \pi \pmod{2\pi}$.

2.3 Massive Thirring-like model with symplectic invariance

2.3.1 Derivation

Using our approach, it is possible to obtain not only evolutionary systems but also non-evolutionary systems with symplectic invariance. To demonstrate this, let us consider the application of the same type of reduction as described in subsections 2.1 and 2.2 to the first negative flows of the matrix DNLS hierarchies. The first negative flow of the matrix Chen–Lee–Liu hierarchy reads as [16, 17]

$$
\begin{cases}
\mathrm{i}q_\tau + m\phi - \frac{1}{2}\phi\chi q = O, \\
\mathrm{i}r_\tau - m\chi + \frac{1}{2}r\phi\chi = O, \\
\mathrm{i}\phi_x + mq - \frac{1}{2}\phi r q = O, \\
\mathrm{i}\chi_x - mr + \frac{1}{2}rq\chi = O,
\end{cases}
\tag{2.31}
$$

while the first negative flow of the matrix Kaup–Newell hierarchy reads as [16, 17]

$$
\begin{cases}
\mathrm{i}q_\tau + m\phi - \frac{1}{2}(q\chi\phi + \phi\chi q) = O, \\
\mathrm{i}r_\tau - m\chi + \frac{1}{2}(r\phi\chi + \chi\phi r) = O, \\
\mathrm{i}\phi_x + mq = O, \\
\mathrm{i}\chi_x - mr = O.
\end{cases}
\tag{2.32}
$$

These are the matrix generalizations of massive Thirring-type models [18–22], where m denotes an arbitrary nonzero constant responsible for the mass terms. The massless limit $m \to 0$ is not considered in this paper; hereafter, the value of m is set as 1, without loss of generality. This is easily achieved by rescaling ϕ, χ, and ∂_τ. The Lax pairs for systems (2.31) and (2.32) are already known [16,17], and thus, they have been omitted here. Both the systems permit the reduction $r = Cq^T$, $\chi = -C\phi^T$, $C^T = -C$, which is the natural extension of the reduction $r = Cq^T$, $C^T = -C$ for the positive flows.

Let us consider a further reduction to the case of row/column vector variables. For the reduced systems to assume a concise form without the imaginary unit and fractions,

we consider the following vector reduction:

$$q=(u_1,\ldots,u_M),\qquad r=2\mathrm{i}Cq^T=2\mathrm{i}\begin{pmatrix}\sum_{k=1}^M C_{1k}u_k\\ \vdots\\ \sum_{k=1}^M C_{Mk}u_k\end{pmatrix},$$

$$\phi=\mathrm{i}(v_1,\ldots,v_M),\qquad \chi=-2\mathrm{i}C\phi^T=2\begin{pmatrix}\sum_{k=1}^M C_{1k}v_k\\ \vdots\\ \sum_{k=1}^M C_{Mk}v_k\end{pmatrix}.$$

Then, because $\phi\chi=0$, system (2.31) reduces to

$$\frac{\partial^2 u_i}{\partial x\partial\tau}+u_i-\Bigg[\sum_{1\le j<k\le M}C_{jk}\Big(\frac{\partial u_j}{\partial\tau}u_k-u_j\frac{\partial u_k}{\partial\tau}\Big)\Bigg]u_i=0,\qquad i=1,2,\ldots,M;\tag{2.33}$$

system (2.32) collapses to

$$\frac{\partial^2 v_i}{\partial\tau\partial x}+v_i-\Bigg[\sum_{1\le j<k\le M}C_{jk}\Big(\frac{\partial v_j}{\partial x}v_k-v_j\frac{\partial v_k}{\partial x}\Big)\Bigg]v_i=0,\qquad i=1,2,\ldots,M,\tag{2.34}$$

where $u_i=\partial v_i/\partial x,\ i=1,2,\ldots,M$. These are considered non-evolutionary symmetries of system (2.7) and system (2.22), respectively. It can be easily observed that (2.33) and (2.34) are equivalent up to a simple interchange of the variables. Nonetheless, the commutativity of the negative and positive flows in each hierarchy itself is of significance. The Lax pair for (2.34) is given by (cf. (2.23))

$$U=\begin{bmatrix}-\lambda & \boldsymbol{v}_x\\ \lambda C\boldsymbol{v}_x^T & O\end{bmatrix},\qquad V=\begin{bmatrix}\frac{1}{\lambda} & -\frac{1}{\lambda}\boldsymbol{v}\\ C\boldsymbol{v}^T & -C\boldsymbol{v}^T\boldsymbol{v}\end{bmatrix},\tag{2.35}$$

where $\boldsymbol{v}=(v_1,v_2,\ldots,v_M)$. In addition, it may be noted that (2.34) can be rewritten as a system for $u_i(=v_{i,x})$. Indeed, (2.34) implies the relation

$$\sum_{1\le i,l\le M}C_{il}\frac{\partial u_i}{\partial\tau}u_l+\Bigg[1-\sum_{1\le j<k\le M}C_{jk}\Big(\frac{\partial v_j}{\partial x}v_k-v_j\frac{\partial v_k}{\partial x}\Big)\Bigg]\sum_{1\le i,l\le M}C_{il}v_iu_l=0,$$

from which we obtain

$$\sum_{1\le j<k\le M}C_{jk}\Big(\frac{\partial v_j}{\partial x}v_k-v_j\frac{\partial v_k}{\partial x}\Big)=\frac{1-\sqrt{1-4\sum_{1\le i,l\le M}C_{il}\frac{\partial u_i}{\partial\tau}u_l}}{2},\tag{2.36}$$

under appropriate boundary conditions. Substituting (2.36) in (2.34) with a simple operation, we arrive at a closed PDE system for u_i,

$$\frac{\partial}{\partial x}\left[\frac{2\frac{\partial u_i}{\partial\tau}}{1+\sqrt{1-4\sum_{1\le j<k\le M}C_{jk}\Big(\frac{\partial u_j}{\partial\tau}u_k-u_j\frac{\partial u_k}{\partial\tau}\Big)}}\right]+u_i=0,\qquad i=1,2,\ldots,M.\tag{2.37}$$

2.3.2 Solution formulas

A set of formulas for the solutions of the matrix massive Thirring model (2.31) with $m = 1$, decaying as $x \to +\infty$, is given by (2.12) supplemented with the following:

$$-\mathrm{i}\phi = J(x,x), \tag{2.38a}$$

$$\mathrm{i}\chi = \bar{J}(x,x), \tag{2.38b}$$

$$J(x,y) = -\int_y^\infty \mathrm{d}s\,\bar{F}(s) - \frac{\mathrm{i}}{2}\int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2\, J(x,s_1)\frac{\partial F(s_1+s_2-x)}{\partial s_2}\bar{F}(s_2+y-x), \quad y \geq x, \tag{2.38c}$$

$$\bar{J}(x,y) = -\int_y^\infty \mathrm{d}s\,F(s) - \frac{\mathrm{i}}{2}\int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2\, \frac{\partial \bar{J}(x,s_1)}{\partial s_1}\frac{\partial \bar{F}(s_1+s_2-x)}{\partial s_2}\int_{s_2+y-x}^\infty \mathrm{d}s_3\,F(s_3), \quad y \geq x. \tag{2.38d}$$

Similarly, a set of formulas for the solutions of the matrix Kaup–Newell-type Thirring model (2.32) with $m = 1$, decaying as $x \to +\infty$, is given by (2.26) together with the following two relations:

$$-\mathrm{i}\phi = \mathcal{K}(x,x), \tag{2.39a}$$

$$\mathrm{i}\chi = \bar{\mathcal{K}}(x,x). \tag{2.39b}$$

In both cases, the *linear* matrix PDEs satisfied by $\bar{F}$ and F are given by

$$\frac{\partial^2 \bar{F}}{\partial\tau\partial x} + \bar{F} = O, \qquad \frac{\partial^2 F}{\partial\tau\partial x} + F = O,$$

and the same relation applies for their primitive functions $\bar{G}$ and G. Applying the same reduction as that in the positive flow case (cf. (2.27)) to the formulas (2.26) and (2.39), we obtain the solution formula for system (2.34),

$$\boldsymbol{v}(x,\tau) = \boldsymbol{k}(x,x;\tau), \tag{2.40a}$$

$$\boldsymbol{k}(x,y) = \boldsymbol{g}(y) - \int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2 \frac{\partial \boldsymbol{k}(x,s_1)}{\partial s_1} C \frac{\partial \boldsymbol{g}(s_1+s_2-x)^T}{\partial s_2}\frac{\partial \boldsymbol{g}(s_2+y-x)}{\partial y}, \quad y \geq x. \tag{2.40b}$$

Here, $\boldsymbol{v} = (v_1, v_2, \ldots, v_M)$, and $\boldsymbol{g}$ solves the linear vector PDE $\boldsymbol{g}_{x\tau} + \boldsymbol{g} = \boldsymbol{0}$. Using the formula (2.40), we can construct the soliton solutions of system (2.34) in a manner similar to that in the positive flow case. In particular, the one-soliton solution of (2.34) is given by

$$\boldsymbol{v}(x,\tau) = \frac{\boldsymbol{b}_1 \mathrm{e}^{\mathrm{i}\xi_1 x + \mathrm{i}\frac{\xi_1}{\xi_1^2+\eta_1^2}\tau} + \boldsymbol{b}_2 \mathrm{e}^{-\mathrm{i}\xi_1 x - \mathrm{i}\frac{\xi_1}{\xi_1^2+\eta_1^2}\tau}}{\cosh\left(\eta_1 x - \frac{\eta_1}{\xi_1^2+\eta_1^2}\tau + \delta\right)}, \qquad \mathrm{i}\xi_1\langle \boldsymbol{b}_1 C, \boldsymbol{b}_2\rangle = \frac{\eta_1^2}{\xi_1^2+\eta_1^2}.$$

The additional conditions $\xi_1 \in \mathbb{R} - \{0\}$, $\eta_1 > 0$, and $\mathrm{e}^{2\delta} \notin \mathbb{R}_{<0}$ guarantee that this solution is regular for real x and τ.

3 Integrable discretizations

In this section, we propose space discretizations of systems (2.7), (2.22), and (2.34) while retaining both the integrability and symplectic invariance in the continuous case. Using a tricky transformation peculiar to the discrete case, we also obtain a novel integrable semi-discretization of the Manakov model (1.1). One-soliton solutions are obtained by solving discrete integral (linear summation) equations of the Gel'fand–Levitan–Marchenko type.

3.1 System of coupled derivative mKdV equations (2.7)

A space-discrete analogue of the (non-reduced) matrix complex mKdV equation (2.10) that is "maximally symmetric" with respect to reduction of the matrix variables is given by [23]

$$\begin{cases} Q_{n,t} + (I_1 - Q_{n+1}R_{n+1})^{-1}Q_{n+1} + (I_1 - Q_nR_n)^{-1}Q_n \\ \qquad - (I_1 - Q_nR_{n+1})^{-1}Q_n - (I_1 - Q_{n-1}R_n)^{-1}Q_{n-1} = O, \\ R_{n,t} + R_{n+1}(I_1 - Q_nR_{n+1})^{-1} + R_n(I_1 - Q_{n-1}R_n)^{-1} \\ \qquad - R_n(I_1 - Q_nR_n)^{-1} - R_{n-1}(I_1 - Q_{n-1}R_{n-1})^{-1} = O. \end{cases} \tag{3.1}$$

Here, "maximally symmetric" implies that (3.1) possibly permits all the interesting reductions corresponding to those in the continuous case. System (3.1) possesses the following Lax pair:

$$L_n = \begin{bmatrix} zI_1 - \left(z + \frac{1}{z}\right) Q_nR_n & \frac{1}{z}Q_n \\ -\left(z + \frac{1}{z}\right) R_n & \frac{1}{z}I_2 \end{bmatrix} = \begin{bmatrix} zI_1 & Q_n \\ O & I_2 \end{bmatrix} \begin{bmatrix} I_1 & O \\ -\left(z + \frac{1}{z}\right) R_n & \frac{1}{z}I_2 \end{bmatrix}, \tag{3.2a}$$

$$M_n = \left[\begin{array}{c|c} -\left(z^2 + \frac{1}{z^2} + 2\right) I_1 & -(I_1 - Q_nR_n)^{-1}Q_n \\ +\left(1 + \frac{1}{z^2}\right)(I_1 - Q_{n-1}R_n)^{-1} & -\frac{1}{z^2}(I_1 - Q_{n-1}R_n)^{-1}Q_{n-1} \\ \hline (z^2 + 1)(I_2 - R_{n-1}Q_{n-1})^{-1}R_{n-1} & \\ +\left(1 + \frac{1}{z^2}\right)(I_2 - R_nQ_{n-1})^{-1}R_n & -\left(1 + \frac{1}{z^2}\right)(I_2 - R_nQ_{n-1})^{-1} \end{array} \right]. \tag{3.2b}$$

Here, z is the spectral parameter independent of n and t; I_1 and I_2 are the $l_1 \times l_1$ and $l_2 \times l_2$ unit matrices, respectively. Indeed, substituting (3.2) in (a space-discrete version of) the zero-curvature condition [42]

$$L_{n,t} + L_nM_n - M_{n+1}L_n = O, \tag{3.3}$$

which is the compatibility condition for the overdetermined linear equations $\Psi_{n+1} = L_n\Psi_n$ and $\Psi_{n,t} = M_n\Psi_n$, we obtain the space-discrete system (3.1). To achieve a discrete counterpart of the reduction (2.11), we first assume the following relations between Q_n $(n \in \mathbb{Z})$ and R_n $(n \in \mathbb{Z})$:

$$R_n = P_n - P_{n-1}, \quad Q_mP_n + Q_nP_m = O, \quad \forall m, n \in \mathbb{Z}. \tag{3.4}$$

If these are satisfied, then system (3.1) is reduced to the form

$$\begin{cases} Q_{n,t} + (I_1 + Q_{n+1}P_n)^{-1}(Q_{n+1} - Q_n) + (I_1 + Q_nP_{n-1})^{-1}(Q_n - Q_{n-1}) = O, \\ \left[P_{n,t} + (P_{n+1} - P_n)(I_1 - Q_nP_{n+1})^{-1} + (P_n - P_{n-1})(I_1 - Q_{n-1}P_n)^{-1}\right] - \left[n \to n-1\right] = O. \end{cases} \tag{3.5}$$

The following choice of Q_n and P_n automatically satisfies (3.4):

$$Q_n = (u_n^{(1)}, \ldots, u_n^{(M)}), \quad P_n = C Q_n^T = \begin{pmatrix} \sum_{k=1}^M C_{1k} u_n^{(k)} \\ \vdots \\ \sum_{k=1}^M C_{Mk} u_n^{(k)} \end{pmatrix}, \quad C^T = -C. \tag{3.6}$$

Substituting (3.6) in (3.5), we obtain an integrable semi-discretization of the coupled derivative mKdV equations (2.7),

$$\frac{\partial u_n^{(i)}}{\partial t} + \frac{u_{n+1}^{(i)} - u_n^{(i)}}{1 + \sum\limits_{1 \le j < k \le M} C_{jk}\left(u_{n+1}^{(j)} u_n^{(k)} - u_n^{(j)} u_{n+1}^{(k)}\right)} + \frac{u_n^{(i)} - u_{n-1}^{(i)}}{1 + \sum\limits_{1 \le j < k \le M} C_{jk}\left(u_n^{(j)} u_{n-1}^{(k)} - u_{n-1}^{(j)} u_n^{(k)}\right)} = 0,$$

$$i = 1, 2, \ldots, M. \tag{3.7}$$

The Lax pair for the reduced system (3.7) is obtained by substituting (3.6) into Q_n and $R_n (= P_n - P_{n-1})$ in (3.2). Moreover, using a gauge transformation, we can restore the ultralocality of the spatial Lax matrix L_n; thus, the Lax pair can be written in the form

$$L_n = \begin{bmatrix} \mu & (\mu+1)\,\boldsymbol{u}_n \\ (\mu-1)\,C\boldsymbol{u}_n^T & I + (\mu+1)\,C\boldsymbol{u}_n^T\boldsymbol{u}_n \end{bmatrix}, \tag{3.8a}$$

$$M_n = \frac{-1}{1 + \langle \boldsymbol{u}_n C, \boldsymbol{u}_{n-1}\rangle} \left[\begin{array}{c|c} \mu - \frac{1}{\mu} & (\mu+1)\,\boldsymbol{u}_n + \left(1 + \frac{1}{\mu}\right)\boldsymbol{u}_{n-1} \\ \hline \begin{array}{c} (\mu-1)\,C\boldsymbol{u}_{n-1}^T \\ + \left(1 - \frac{1}{\mu}\right) C\boldsymbol{u}_n^T \end{array} & \begin{array}{c} (\mu+1)\,C\boldsymbol{u}_{n-1}^T\boldsymbol{u}_n \\ + \left(1 + \frac{1}{\mu}\right) C\boldsymbol{u}_n^T\boldsymbol{u}_{n-1} \end{array} \end{array}\right], \tag{3.8b}$$

where $\boldsymbol{u}_n = (u_n^{(1)}, u_n^{(2)}, \ldots, u_n^{(M)})$ is a row vector and $\mu := z^2$ is the "new" spectral parameter. It is easy to see that the discrete eigenvalue problem $\Psi_{n+1} = L_n \Psi_n$ with (3.8a) reduces to the continuous eigenvalue problem $\Psi_x = U\Psi$ with (2.8a) in a suitable continuous limit (cf. ref. 43).

3.2 Manakov model (1.1)

Let us consider the canonical case of the coupling constants (cf. the introduction) in (3.7), namely, $C_{2j-1\,2k} = -C_{2k\,2j-1} = \delta_{jk}$, $C_{2j-1\,2k-1} = C_{2j\,2k} = 0$, and $M = 2m$, and change the dependent variables as follows:

$$u_n^{(2j-1)} =: (-\mathrm{i})^n \mathrm{e}^{2\mathrm{i}t} q_n^{(j)}, \quad u_n^{(2j)} =: \mathrm{i}^{n+1} \mathrm{e}^{-2\mathrm{i}t} r_n^{(j)}, \quad j = 1, 2, \ldots, m.$$

Then, system (3.7) is converted into the following form:

$$\begin{cases} \mathrm{i}\dfrac{\partial q_n^{(j)}}{\partial t} + \dfrac{q_{n+1}^{(j)} - \mathrm{i} q_n^{(j)}}{1 + \sum\limits_{k=1}^m \left(q_{n+1}^{(k)} r_n^{(k)} + q_n^{(k)} r_{n+1}^{(k)}\right)} + \dfrac{q_{n-1}^{(j)} + \mathrm{i} q_n^{(j)}}{1 + \sum\limits_{k=1}^m \left(q_n^{(k)} r_{n-1}^{(k)} + q_{n-1}^{(k)} r_n^{(k)}\right)} - 2q_n^{(j)} = 0, \\ \mathrm{i}\dfrac{\partial r_n^{(j)}}{\partial t} - \dfrac{r_{n+1}^{(j)} + \mathrm{i} r_n^{(j)}}{1 + \sum\limits_{k=1}^m \left(q_{n+1}^{(k)} r_n^{(k)} + q_n^{(k)} r_{n+1}^{(k)}\right)} - \dfrac{r_{n-1}^{(j)} - \mathrm{i} r_n^{(j)}}{1 + \sum\limits_{k=1}^m \left(q_n^{(k)} r_{n-1}^{(k)} + q_{n-1}^{(k)} r_n^{(k)}\right)} + 2r_n^{(j)} = 0, \\ \hfill j = 1, 2, \ldots, m. \end{cases}$$

By further imposing the reduction $r_n^{(j)} = \sigma_j q_n^{(j)*}$, $\sigma_j = \pm 1$, we obtain a new integrable semi-discretization of the system of coupled NLS equations. The simplest case $r_n^{(j)} = -q_n^{(j)*}$ $(j = 1, 2, \ldots, m)$ provides a space-discrete analogue of the $U(m)$-invariant Manakov model (1.1),

$$\mathrm{i}\boldsymbol{q}_{n,t} + \frac{\boldsymbol{q}_{n+1} - \mathrm{i}\boldsymbol{q}_n}{1 - \langle \boldsymbol{q}_{n+1}, \boldsymbol{q}_n^* \rangle - \langle \boldsymbol{q}_n, \boldsymbol{q}_{n+1}^* \rangle} + \frac{\boldsymbol{q}_{n-1} + \mathrm{i}\boldsymbol{q}_n}{1 - \langle \boldsymbol{q}_n, \boldsymbol{q}_{n-1}^* \rangle - \langle \boldsymbol{q}_{n-1}, \boldsymbol{q}_n^* \rangle} - 2\boldsymbol{q}_n = \boldsymbol{0}, \tag{3.9}$$

where $\boldsymbol{q}_n = \left(q_n^{(1)}, q_n^{(2)}, \ldots, q_n^{(m)}\right)$.

3.3 Solutions to system (3.7)

A set of formulas for the solutions of the space-discrete matrix complex mKdV system (3.1) as well as its commuting flows, which tend to zero as $n \to +\infty$, is given by

$$Q_n = K(n,n) + K(n+1,n+1), \tag{3.10a}$$

$$R_n = \bar{K}(n,n) + \bar{K}(n+1,n+1), \tag{3.10b}$$

$$\begin{aligned} K(n,m) = \bar{G}(m) + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} &[K(n,n+j) + K(n,n+j+1)]\,[G(n+j+k+1) + G(n+j+k+2)] \\ &\times \left[\bar{G}(m+k) + \bar{G}(m+k+1)\right], \quad m \geq n, \end{aligned} \tag{3.10c}$$

$$\begin{aligned} \bar{K}(n,m) = -G(m) + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} &\left[\bar{K}(n,n+j) + \bar{K}(n,n+j+1)\right]\,\left[\bar{G}(n+j+k) + \bar{G}(n+j+k+1)\right] \\ &\times [G(m+k) + G(m+k+1)], \quad m \geq n. \end{aligned} \tag{3.10d}$$

Here, the functions $\bar{G}(n)$ and $G(n)$ satisfy the corresponding *linear* uncoupled system of matrix differential-difference equations, *e.g.*,

$$\frac{\partial \bar{G}(n)}{\partial t} + \bar{G}(n+1) - \bar{G}(n-1) = O, \qquad \frac{\partial G(n)}{\partial t} + G(n+1) - G(n-1) = O$$

for the flow (3.1), and decay rapidly as $n \to +\infty$. The reduction given by (3.4) and (3.6) is achieved at the level of the solution formulas by setting

$$\bar{G}(n) = (g_1, g_2, \ldots, g_M)(n) =: \boldsymbol{g}(n), \qquad G(n) = -C\left[\bar{G}(n) - \bar{G}(n-1)\right]^T.$$

With this reduction, the set of solution formulas (3.10) is reduced to a compact form. Thus, the solutions to the semi-discrete coupled derivative mKdV equations (3.7), decaying as $n \to +\infty$, can be constructed from those of the linear vector differential-difference equation $\partial \boldsymbol{g}(n)/\partial t + \boldsymbol{g}(n+1) - \boldsymbol{g}(n-1) = \boldsymbol{0}$ through the formula

$$\boldsymbol{u}_n(t) = \boldsymbol{k}(n,n;t) + \boldsymbol{k}(n+1,n+1;t), \tag{3.11a}$$

$$\begin{aligned} \boldsymbol{k}(n,m) = \boldsymbol{g}(m) + \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} &\left[\boldsymbol{k}(n,n+j) + \boldsymbol{k}(n,n+j+1)\right] C \left[\boldsymbol{g}(n+j+l) - \boldsymbol{g}(n+j+l+2)\right]^T \\ &\times \left[\boldsymbol{g}(m+l) + \boldsymbol{g}(m+l+1)\right], \quad m \geq n. \end{aligned} \tag{3.11b}$$

Here, $\boldsymbol{u}_n = (u_n^{(1)}, u_n^{(2)}, \ldots, u_n^{(M)})$ and $\boldsymbol{k}(n,m)$ are M-component row vectors. Substituting the expressions

$$\boldsymbol{g}(n,t) = \boldsymbol{a}_1 \mu_1^{-n} \mathrm{e}^{(\mu_1-\mu_1^{-1})t} + \boldsymbol{a}_2 \mu_2^{-n} \mathrm{e}^{(\mu_2-\mu_2^{-1})t}, \quad |\mu_j| > 1 \ (j=1,2), \quad \mu_1 \neq \mu_2, \quad \langle \boldsymbol{a}_1 C, \boldsymbol{a}_2 \rangle \neq 0,$$
$$\boldsymbol{k}(n,m;t) = \boldsymbol{k}_1(n,t) \mu_1^{-m} \mathrm{e}^{(\mu_1-\mu_1^{-1})t} + \boldsymbol{k}_2(n,t) \mu_2^{-m} \mathrm{e}^{(\mu_2-\mu_2^{-1})t}$$

into (3.11) and solving it with respect to $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, we obtain the "unrefined" one-soliton solution of system (3.7) in the additive form

$$\boldsymbol{u}_n(t) = \boldsymbol{p}_n(t) + \boldsymbol{p}_{n+1}(t), \tag{3.12}$$

where $\boldsymbol{p}_n(t)$ is given by

$$\begin{aligned} \boldsymbol{p}_n(t) &= \boldsymbol{k}_1(n,t) \mu_1^{-n} \mathrm{e}^{(\mu_1-\mu_1^{-1})t} + \boldsymbol{k}_2(n,t) \mu_2^{-n} \mathrm{e}^{(\mu_2-\mu_2^{-1})t} \\ &= \frac{\boldsymbol{a}_1 \mu_1^{-n} \mathrm{e}^{(\mu_1-\mu_1^{-1})t} + \boldsymbol{a}_2 \mu_2^{-n} \mathrm{e}^{(\mu_2-\mu_2^{-1})t}}{1 + \frac{(\mu_1-\mu_2)(1+\mu_1)(1+\mu_2)}{(1-\mu_1\mu_2)^2} \langle \boldsymbol{a}_1 C, \boldsymbol{a}_2 \rangle \mu_1^{-n} \mu_2^{-n} \mathrm{e}^{(\mu_1+\mu_2-\mu_1^{-1}-\mu_2^{-1})t}}. \end{aligned} \tag{3.13}$$

In fact, the row vector $\boldsymbol{p}_n(t)$ itself satisfies the nonlinear differential-difference equation (cf. (3.7) and (3.12)),

$$\frac{\partial \boldsymbol{p}_n}{\partial t} + \frac{\boldsymbol{p}_{n+1} - \boldsymbol{p}_{n-1}}{1 + \langle \boldsymbol{p}_{n+1} C, \boldsymbol{p}_n \rangle + \langle \boldsymbol{p}_n C, \boldsymbol{p}_{n-1} \rangle + \langle \boldsymbol{p}_{n+1} C, \boldsymbol{p}_{n-1} \rangle} = \boldsymbol{0}. \tag{3.14}$$

Note that the denominator in the expression (3.13) may become zero for certain values of n and t. By introducing a new parametrization,

$$\frac{(\mu_1-\mu_2)(1+\mu_1)(1+\mu_2)}{(1-\mu_1\mu_2)^2} \langle \boldsymbol{a}_1 C, \boldsymbol{a}_2 \rangle =: \mathrm{e}^{-2\delta} \ \ (\delta \in \mathbb{C}),$$
$$\boldsymbol{a}_1 =: 2\mathrm{e}^{-\delta} \boldsymbol{b}_1, \quad \boldsymbol{a}_2 =: 2\mathrm{e}^{-\delta} \boldsymbol{b}_2, \quad \mu_1 = \mathrm{e}^{\alpha - \mathrm{i}\beta}, \quad \mu_2 = \mathrm{e}^{\alpha + \mathrm{i}\beta},$$

(3.13) can be rewritten as

$$\boldsymbol{p}_n(t) = \frac{\boldsymbol{b}_1 \mathrm{e}^{\mathrm{i}\beta n - 2\mathrm{i}(\cosh\alpha \, \sin\beta)t} + \boldsymbol{b}_2 \mathrm{e}^{-\mathrm{i}\beta n + 2\mathrm{i}(\cosh\alpha \, \sin\beta)t}}{\cosh\left[\alpha n - 2(\sinh\alpha \, \cos\beta)t + \delta\right]}, \tag{3.15}$$

with the condition $-4\mathrm{i}(\cosh\alpha + \cos\beta)\sin\beta \, \langle \boldsymbol{b}_1 C, \boldsymbol{b}_2 \rangle = (\sinh\alpha)^2$. The "one-soliton" solution (3.15) of (3.14) resembles the soliton solution of the semi-discrete vector mKdV equation (or, the vector modified Volterra lattice) $\partial \boldsymbol{q}_n / \partial t = \big(1 + \langle \boldsymbol{q}_n, \boldsymbol{q}_n \rangle\big)(\boldsymbol{q}_{n+1} - \boldsymbol{q}_{n-1})$ given by

$$\boldsymbol{q}_n(t) = \frac{\boldsymbol{c}_1 \mathrm{e}^{\mathrm{i}\beta n + 2\mathrm{i}(\cosh\alpha \, \sin\beta)t} + \boldsymbol{c}_2 \mathrm{e}^{-\mathrm{i}\beta n - 2\mathrm{i}(\cosh\alpha \, \sin\beta)t}}{\cosh\left[\alpha n + 2(\sinh\alpha \, \cos\beta)t + \delta\right]},$$

under the conditions $\langle \boldsymbol{c}_1, \boldsymbol{c}_1 \rangle = \langle \boldsymbol{c}_2, \boldsymbol{c}_2 \rangle = 0$ and $2\langle \boldsymbol{c}_1, \boldsymbol{c}_2 \rangle = (\sinh\alpha)^2$. Moreover, if the "reality conditions" $\alpha > 0$, $0 < \beta < \pi$ (or $-\pi < \beta < 0$), and $\mathrm{e}^{2\delta} \notin \mathbb{R}_{<0}$ are imposed, (3.15) provides the bright one-soliton solution that is indeed regular for real n and t. Owing to the discrete nature of the space variable n, there can exist other cases wherein the solution is regular, *e.g.*, $\alpha > 0$, $\beta = \pi/2$, and $\mathrm{e}^{2\alpha n + 2\delta} \neq -1$, $\forall n$.

3.4 System of coupled derivative mKdV equations (2.22)

In this subsection and the subsequent subsections, we use the forward difference operator $\boldsymbol{\Delta}_n$ to indicate

$$\boldsymbol{\Delta}_n f_{n+j} := f_{n+j+1} - f_{n+j}.$$

It can be recalled that an integrable semi-discretization of the third-order matrix Kaup–Newell system (2.19) is given by [23]

$$\begin{cases} q_{n,t} + \boldsymbol{\Delta}_n \big[(I_1 - q_n r_n)^{-1} q_n + (I_1 + q_{n-1} r_n)^{-1} q_{n-1} \big] = O, \\ r_{n,t} + \boldsymbol{\Delta}_n \big[r_n (I_1 + q_{n-1} r_n)^{-1} + r_{n-1} (I_1 - q_{n-1} r_{n-1})^{-1} \big] = O. \end{cases} \tag{3.16}$$

This system possesses the following Lax pair:

$$L_n = \begin{bmatrix} zI_1 - \left(z - \frac{1}{z}\right) q_n r_n & \frac{1}{z} q_n \\ \left(-z + \frac{1}{z}\right) r_n & \frac{1}{z} I_2 \end{bmatrix} = \begin{bmatrix} zI_1 & q_n \\ O & I_2 \end{bmatrix} \begin{bmatrix} I_1 & O \\ \left(-z + \frac{1}{z}\right) r_n & \frac{1}{z} I_2 \end{bmatrix}, \tag{3.17a}$$

$$M_n = \left[\begin{array}{c|c} \begin{array}{c} \left[-\left(z^2 - 1\right) + \left(1 - \frac{1}{z^2}\right) \right] I_1 \\ - \left(1 - \frac{1}{z^2}\right) (I_1 + q_{n-1} r_n)^{-1} \end{array} & \begin{array}{c} -(I_1 - q_n r_n)^{-1} q_n \\ - \frac{1}{z^2} (I_1 + q_{n-1} r_n)^{-1} q_{n-1} \end{array} \\ \hline \begin{array}{c} \left(z^2 - 1\right) (I_2 - r_{n-1} q_{n-1})^{-1} r_{n-1} \\ + \left(1 - \frac{1}{z^2}\right) (I_2 + r_n q_{n-1})^{-1} r_n \end{array} & \left(1 - \frac{1}{z^2}\right) (I_2 + r_n q_{n-1})^{-1} \end{array} \right]. \tag{3.17b}$$

Indeed, the substitution of (3.17) in the zero-curvature condition (3.3) gives the space-discrete system (3.16). As system (2.19) permits the reduction $r \propto Cq^T$, $C^T = -C$, so system (3.16) allows the corresponding reduction $r_n = Cq^T_{n-\frac{1}{2}}$, $C^T = -C$. In particular, considering the vector reduction

$$q_n = (u_n^{(1)}, \ldots, u_n^{(M)}), \quad r_n = Cq^T_{n-\frac{1}{2}}, \quad C^T = -C,$$

we obtain an integrable semi-discretization of the coupled derivative mKdV equations (2.22),

$$\begin{aligned} \frac{\partial u_n^{(i)}}{\partial t} + \boldsymbol{\Delta}_n \Bigg[& \frac{u_n^{(i)}}{1 - \displaystyle\sum_{1 \le j < k \le M} C_{jk} \big(u_n^{(j)} u_{n-\frac{1}{2}}^{(k)} - u_{n-\frac{1}{2}}^{(j)} u_n^{(k)} \big)} \\ & + \frac{u_{n-1}^{(i)}}{1 - \displaystyle\sum_{1 \le j < k \le M} C_{jk} \big(u_{n-\frac{1}{2}}^{(j)} u_{n-1}^{(k)} - u_{n-1}^{(j)} u_{n-\frac{1}{2}}^{(k)} \big)} \Bigg] = 0, \quad i = 1, 2, \ldots, M. \end{aligned} \tag{3.18}$$

This space difference scheme depends on the five points: n, $n \pm \frac{1}{2}$, and $n \pm 1$. In fact, we can derive a simpler, three-point difference scheme for (2.22) from the same matrix system (3.16). For this purpose, we consider the vector reduction

$$q_n = (u_n^{(1)}, \ldots, u_n^{(M)}), \quad r_n = C(q_n + q_{n-1})^T, \quad C^T = -C. \tag{3.19}$$

Then, considering the relation $q_n C q_m^T + q_m C q_n^T = 0$, we can observe that (3.16) is reduced to an alternative lattice version of (2.22),

$$\frac{\partial u_n^{(i)}}{\partial t} + \boldsymbol{\Delta}_n \left[\frac{u_n^{(i)} + u_{n-1}^{(i)}}{1 - \sum_{1 \le j < k \le M} C_{jk} \big(u_n^{(j)} u_{n-1}^{(k)} - u_{n-1}^{(j)} u_n^{(k)} \big)} \right] = 0, \quad i = 1, 2, \dots, M. \tag{3.20}$$

System (3.20) resembles the integrable semi-discretization of the vector third-order Heisenberg ferromagnet model (1.3) (see (2.22) in ref. 7 or (5.25) in ref. 44). The Lax pair for (3.20) is obtained by substituting (3.19) into q_n and r_n in (3.17). Moreover, using a gauge transformation, we can restore the ultralocality of the L_n-matrix; thus, the Lax pair can be written in the form

$$L_n = \begin{bmatrix} \mu & (\mu+1)\, \boldsymbol{u}_n \\ (-\mu+1)\, C\boldsymbol{u}_n^T & I - (\mu-1)\, C\boldsymbol{u}_n^T \boldsymbol{u}_n \end{bmatrix}, \tag{3.21a}$$

$$M_n = \frac{1}{1 - \langle \boldsymbol{u}_n C, \boldsymbol{u}_{n-1} \rangle} \left[\begin{array}{c|c} -\mu + \frac{1}{\mu} & -(\mu+1)\, \boldsymbol{u}_n - \left(1 + \frac{1}{\mu}\right) \boldsymbol{u}_{n-1} \\ \hline \begin{array}{c} (\mu-1)\, C\boldsymbol{u}_{n-1}^T \\ + \left(1 - \frac{1}{\mu}\right) C\boldsymbol{u}_n^T \end{array} & \begin{array}{c} (\mu-1)\, C\boldsymbol{u}_{n-1}^T \boldsymbol{u}_n \\ - \left(1 - \frac{1}{\mu}\right) C\boldsymbol{u}_n^T \boldsymbol{u}_{n-1} \end{array} \end{array} \right], \tag{3.21b}$$

where $\boldsymbol{u}_n = (u_n^{(1)}, u_n^{(2)}, \dots, u_n^{(M)})$ and $\mu := z^2$. The discrete eigenvalue problem $\Psi_{n+1} = L_n \Psi_n$ with (3.21a) reduces to the continuous eigenvalue problem $\Psi_x = U\Psi$ with (2.23a) in a suitable continuous limit (cf. ref. 43).

It should be noted that the two systems (3.7) and (3.20) are connected through the change of variables

$$u_n^{(i)} \to (-1)^n u_n^{(i)}, \quad t \to -t.$$

This correspondence is not surprising as their respective "ancestor" systems (3.1) and (3.16) are connected through the same type of transformation [23].

3.5 Massive Thirring-like model (2.34)

A space discretization of the matrix massive Thirring-type model (2.32),

$$\begin{cases} q_{n,\tau} + \mathrm{i}(\phi_n + \phi_{n+1}) + 2(q_n \chi_{n+1} \phi_n + \phi_{n+1} \chi_{n+1} q_n) = O, \\ r_{n,\tau} - \mathrm{i}(\chi_n + \chi_{n+1}) - 2(r_n \phi_n \chi_n + \chi_{n+1} \phi_n r_n) = O, \\ \phi_n - \phi_{n+1} + \mathrm{i} q_n = O, \\ \chi_n - \chi_{n+1} - \mathrm{i} r_n = O, \end{cases} \tag{3.22}$$

together with its Lax pair, was proposed in ref. 23. This is considered the first negative flow of the semi-discrete Kaup–Newell hierarchy, which contains (3.16) as a positive flow. It should be noted that system (3.22) permits the reduction $r_n = C q_{n-\frac{1}{2}}^T$, $\chi_n = -C \phi_{n-\frac{1}{2}}^T$, $C^T = -C$. In particular, the vector reduction

$$q_n = (u_n^{(1)}, \dots, u_n^{(M)}), \quad r_n = C q_{n-\frac{1}{2}}^T,$$
$$\phi_n = \mathrm{i}(v_n^{(1)}, \dots, v_n^{(M)}), \quad \chi_n = -C \phi_{n-\frac{1}{2}}^T, \quad C^T = -C$$

simplifies (3.22) to a single vector equation,

$$\frac{\partial}{\partial\tau}(v_n^{(i)} - v_{n+1}^{(i)}) + v_n^{(i)} + v_{n+1}^{(i)} - 2\Bigg[\sum_{1\le j<k\le M} C_{jk}(v_{n+1}^{(j)} v_{n+\frac{1}{2}}^{(k)} - v_{n+\frac{1}{2}}^{(j)} v_{n+1}^{(k)})\Bigg] v_{n+1}^{(i)}$$
$$- 2\Bigg[\sum_{1\le j<k\le M} C_{jk}(v_{n+\frac{1}{2}}^{(j)} v_n^{(k)} - v_n^{(j)} v_{n+\frac{1}{2}}^{(k)})\Bigg] v_n^{(i)} = 0, \quad i = 1, 2, \ldots, M, \tag{3.23}$$

with $u_n^{(i)} = \mathbf{\Delta}_n v_n^{(i)}$. This is a three-point (n, $n+\frac{1}{2}$, $n+1$) difference scheme for the massive Thirring-like model (2.34), and is considered a non-evolutionary potential symmetry of system (3.18). In the same manner as described in subsection 3.4, we can also obtain a simpler, two-point difference scheme for (2.34) from the matrix system (3.22). Indeed, if we set

$$\begin{aligned} &q_n = (u_n^{(1)}, \ldots, u_n^{(M)}), \quad r_n = C(q_n + q_{n-1})^T, \\ &\phi_n = \mathrm{i}(v_n^{(1)}, \ldots, v_n^{(M)}), \quad \chi_n = -C(\phi_n + \phi_{n-1})^T, \quad C^T = -C, \end{aligned} \tag{3.24}$$

and utilize the relation $\phi_n C \phi_m^T + \phi_m C \phi_n^T = 0$, (3.22) is reduced to an alternative space discretization of (2.34),

$$\frac{\partial}{\partial\tau}(v_n^{(i)} - v_{n+1}^{(i)}) + v_n^{(i)} + v_{n+1}^{(i)} - 2\Bigg[\sum_{1\le j<k\le M} C_{jk}(v_{n+1}^{(j)} v_n^{(k)} - v_n^{(j)} v_{n+1}^{(k)})\Bigg](v_n^{(i)} + v_{n+1}^{(i)}) = 0,$$
$$i = 1, 2, \ldots, M, \tag{3.25}$$

where $u_n^{(i)} = \mathbf{\Delta}_n v_n^{(i)} (= v_{n+1}^{(i)} - v_n^{(i)})$. This is considered a non-evolutionary symmetry of (the potential form of) system (3.20). The Lax pair for (3.25) is given by (cf. (3.21) and (2.35))

$$L_n = \begin{bmatrix} \mu & (\mu+1)(\boldsymbol{v}_{n+1} - \boldsymbol{v}_n) \\ (-\mu+1)\, C\left(\boldsymbol{v}_{n+1}^T - \boldsymbol{v}_n^T\right) & I - (\mu-1)\, C\left(\boldsymbol{v}_{n+1}^T - \boldsymbol{v}_n^T\right)(\boldsymbol{v}_{n+1} - \boldsymbol{v}_n) \end{bmatrix}, \tag{3.26a}$$

$$M_n = \frac{1}{\mu - 1}\begin{bmatrix} \mu+1 & -2(\mu+1)\,\boldsymbol{v}_n \\ -2(\mu-1)\, C\boldsymbol{v}_n^T & 4(\mu-1)\, C\boldsymbol{v}_n^T \boldsymbol{v}_n \end{bmatrix}, \tag{3.26b}$$

where $\boldsymbol{v}_n = (v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(M)})$. Along parallel lines with the continuous case (cf. subsection 2.3.1), we can rewrite (3.25) as a closed differential-difference system for $u_n^{(i)}$. The resultant system provides an integrable semi-discretization of system (2.37).

3.6 Solutions to systems (3.20) and (3.25)

In analogy with the continuous case (cf. (2.26)), a set of formulas for the solutions of the space-discrete matrix Kaup–Newell system (3.16) as well as its commuting flows, which

tend to zero as $n \to +\infty$, can be presented in the following difference form [26]:

$$q_n = \mathbf{\Delta}_n \mathcal{K}(n,n), \tag{3.27a}$$

$$r_n = \mathbf{\Delta}_n \bar{\mathcal{K}}(n,n), \tag{3.27b}$$

$$\begin{aligned}\mathcal{K}(n,m) &= -\sum_{s=m}^{\infty} \bar{F}(s) + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \left[\mathcal{K}(n,n+j) - \mathcal{K}(n,n+j+1)\right] F(n+j+k+1)\bar{F}(m+k) \\ &= \bar{G}(m) + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \left[\mathcal{K}(n,n+j) - \mathcal{K}(n,n+j+1)\right]\left[G(n+j+k+1) - G(n+j+k+2)\right] \\ &\qquad \times \left[\bar{G}(m+k) - \bar{G}(m+k+1)\right], \quad m \geq n,\end{aligned} \tag{3.27c}$$

$$\begin{aligned}\bar{\mathcal{K}}(n,m) &= -\sum_{s=m}^{\infty} F(s) - \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \left[\bar{\mathcal{K}}(n,n+j) - \bar{\mathcal{K}}(n,n+j+1)\right] \bar{F}(n+j+k)F(m+k) \\ &= G(m) - \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \left[\bar{\mathcal{K}}(n,n+j) - \bar{\mathcal{K}}(n,n+j+1)\right]\left[\bar{G}(n+j+k) - \bar{G}(n+j+k+1)\right] \\ &\qquad \times \left[G(m+k) - G(m+k+1)\right], \quad m \geq n.\end{aligned} \tag{3.27d}$$

Here, the functions $\bar{F}(n)$ and $F(n)$ satisfy the corresponding *linear* uncoupled system of matrix differential-difference equations, *e.g.*,

$$\frac{\partial \bar{F}(n)}{\partial t} + \bar{F}(n+1) - \bar{F}(n-1) = O, \qquad \frac{\partial F(n)}{\partial t} + F(n+1) - F(n-1) = O \tag{3.28}$$

for the flow (3.16), and decay rapidly as $n \to +\infty$. The matrices $\bar{G}(n)$ and $G(n)$ are the "primitive functions" of $\bar{F}(n)$ and $F(n)$, respectively, that also decay as $n \to +\infty$ and satisfy the same linear system, that is, $\bar{G}(n) := -\sum_{s=n}^{\infty} \bar{F}(s)$ and $G(n) := -\sum_{s=n}^{\infty} F(s)$. The reduction (3.19) is realized at the level of the solution formulas (3.27) by setting

$$\bar{G}(n) = (g_1, g_2, \ldots, g_M)(n) =: \boldsymbol{g}(n), \qquad G(n) = C\left[\bar{G}(n) + \bar{G}(n-1)\right]^T. \tag{3.29}$$

In particular, the solutions to the semi-discrete coupled derivative mKdV equations (3.20), decaying as $n \to +\infty$, can be constructed from those of the linear vector differential-difference equation $\partial \boldsymbol{g}(n)/\partial t + \boldsymbol{g}(n+1) - \boldsymbol{g}(n-1) = \mathbf{0}$ through the compact formula

$$\boldsymbol{u}_n(t) = \mathbf{\Delta}_n \boldsymbol{k}(n,n;t), \tag{3.30a}$$

$$\begin{aligned}\boldsymbol{k}(n,m) &= \boldsymbol{g}(m) + \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \left[\boldsymbol{k}(n,n+j) - \boldsymbol{k}(n,n+j+1)\right] C \left[\boldsymbol{g}(n+j+l) - \boldsymbol{g}(n+j+l+2)\right]^T \\ &\qquad \times \left[\boldsymbol{g}(m+l) - \boldsymbol{g}(m+l+1)\right], \quad m \geq n.\end{aligned} \tag{3.30b}$$

Here, $\boldsymbol{u}_n = (u_n^{(1)}, u_n^{(2)}, \ldots, u_n^{(M)})$ and $\boldsymbol{k}(n,m)$ are M-component row vectors. Substituting the expressions

$$\boldsymbol{g}(n,t) = \boldsymbol{a}_1 \mu_1^{-n} \mathrm{e}^{(\mu_1 - \mu_1^{-1})t} + \boldsymbol{a}_2 \mu_2^{-n} \mathrm{e}^{(\mu_2 - \mu_2^{-1})t}, \quad |\mu_j| > 1 \ (j = 1,2), \quad \mu_1 \neq \mu_2, \quad \langle \boldsymbol{a}_1 C, \boldsymbol{a}_2 \rangle \neq 0,$$
$$\boldsymbol{k}(n,m;t) = \boldsymbol{k}_1(n,t)\mu_1^{-m} \mathrm{e}^{(\mu_1 - \mu_1^{-1})t} + \boldsymbol{k}_2(n,t)\mu_2^{-m} \mathrm{e}^{(\mu_2 - \mu_2^{-1})t}$$

into (3.30) and solving it with respect to $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, we obtain the "unrefined" one-soliton solution of system (3.20) in the difference form

$$\boldsymbol{u}_n(t) = \boldsymbol{\Delta}_n \left\{ \frac{\boldsymbol{a}_1 \mu_1^{-n} \mathrm{e}^{(\mu_1 - \mu_1^{-1})t} + \boldsymbol{a}_2 \mu_2^{-n} \mathrm{e}^{(\mu_2 - \mu_2^{-1})t}}{1 - \frac{(\mu_1-\mu_2)(1-\mu_1)(1-\mu_2)}{(1-\mu_1\mu_2)^2} \langle \boldsymbol{a}_1 C, \boldsymbol{a}_2 \rangle \mu_1^{-n} \mu_2^{-n} \mathrm{e}^{(\mu_1+\mu_2-\mu_1^{-1}-\mu_2^{-1})t}} \right\}. \tag{3.31}$$

Note that the denominator in the expression (3.31) may become zero for certain values of n and t. By introducing a new parametrization,

$$\begin{gathered} -\frac{(\mu_1-\mu_2)(1-\mu_1)(1-\mu_2)}{(1-\mu_1\mu_2)^2} \langle \boldsymbol{a}_1 C, \boldsymbol{a}_2 \rangle =: \mathrm{e}^{-2\delta} \;\; (\delta \in \mathbb{C}), \\ \boldsymbol{a}_1 =: 2\mathrm{e}^{-\delta} \boldsymbol{b}_1, \quad \boldsymbol{a}_2 =: 2\mathrm{e}^{-\delta} \boldsymbol{b}_2, \quad \mu_1 = \mathrm{e}^{\alpha - \mathrm{i}\beta}, \quad \mu_2 = \mathrm{e}^{\alpha + \mathrm{i}\beta}, \end{gathered} \tag{3.32}$$

(3.31) can be rewritten as

$$\begin{aligned} \boldsymbol{u}_n(t) &= \boldsymbol{\Delta}_n \left\{ \frac{\boldsymbol{b}_1 \mathrm{e}^{\mathrm{i}\beta n - 2\mathrm{i}(\cosh\alpha \sin\beta)t} + \boldsymbol{b}_2 \mathrm{e}^{-\mathrm{i}\beta n + 2\mathrm{i}(\cosh\alpha\sin\beta)t}}{\cosh\left[\alpha n - 2(\sinh\alpha\cos\beta)t + \delta\right]} \right\} \\ &= \frac{2\mathrm{i}\sqrt{\sin\left(\frac{\beta+\mathrm{i}\alpha}{2}\right)\sin\left(\frac{\beta-\mathrm{i}\alpha}{2}\right)}}{\cosh^2\left[\alpha n - 2(\sinh\alpha\cos\beta)t + \delta'\right] + \sinh^2\left(\frac{\alpha}{2}\right)} \Big\{ \boldsymbol{b}_1' \cosh\left[\alpha n - 2(\sinh\alpha\cos\beta)t + \delta' + \mathrm{i}\varphi\right] \\ &\quad \times \mathrm{e}^{\mathrm{i}\beta n - 2\mathrm{i}(\cosh\alpha\sin\beta)t} - \boldsymbol{b}_2' \cosh\left[\alpha n - 2(\sinh\alpha\cos\beta)t + \delta' - \mathrm{i}\varphi\right] \mathrm{e}^{-\mathrm{i}\beta n + 2\mathrm{i}(\cosh\alpha\sin\beta)t} \Big\}, \end{aligned} \tag{3.33}$$

with the condition $4\mathrm{i}(\cosh\alpha - \cos\beta)\sin\beta \,\langle \boldsymbol{b}_1 C, \boldsymbol{b}_2 \rangle = (\sinh\alpha)^2$. The constant φ on the right-hand side of (3.33) is defined as

$$\exp(\mathrm{i}\varphi) := \frac{\sin\left(\frac{\beta+\mathrm{i}\alpha}{2}\right)}{\sqrt{\sin\left(\frac{\beta+\mathrm{i}\alpha}{2}\right)\sin\left(\frac{\beta-\mathrm{i}\alpha}{2}\right)}},$$

and the new shifted parameters δ', $\boldsymbol{b}_1'$, and $\boldsymbol{b}_2'$ are given by $\delta' := \delta + \frac{\alpha}{2}$, $\boldsymbol{b}_1' := \mathrm{e}^{\mathrm{i}\frac{\beta}{2}} \boldsymbol{b}_1$, and $\boldsymbol{b}_2' := \mathrm{e}^{-\mathrm{i}\frac{\beta}{2}} \boldsymbol{b}_2$. If we impose the "reality conditions" $\alpha > 0$, $0 < \beta < \pi$ (or $-\pi < \beta < 0$), and $\mathrm{e}^{2\delta} \notin \mathbb{R}_{<0}$, (3.33) provides the bright one-soliton solution of (3.20) that is indeed regular for real n and t.

A set of formulas for the solutions of the first negative flow (3.22) of the semi-discrete matrix Kaup–Newell hierarchy, decaying as $n \to +\infty$, is completed by supplementing (3.27) with the following:

$$-\mathrm{i}\phi_n = \mathcal{K}(n,n), \tag{3.34a}$$

$$\mathrm{i}\chi_n = \bar{\mathcal{K}}(n,n). \tag{3.34b}$$

The *linear* uncoupled system of matrix differential-difference equations to be satisfied by $\bar{F}(n)$ and $F(n)$ in this flow reads as

$$\frac{\partial \bar{F}(n)}{\partial \tau} - \frac{\partial \bar{F}(n+1)}{\partial \tau} + \bar{F}(n) + \bar{F}(n+1) = O, \qquad \frac{\partial F(n)}{\partial \tau} - \frac{\partial F(n+1)}{\partial \tau} + F(n) + F(n+1) = O,$$

and the same relation applies for their “primitive functions” $\bar{G}(n)$ and $G(n)$. Assuming the same restriction as that in the positive flow case (cf. (3.29)), we can realize the reduction (3.24) on the solution formulas (3.27) and (3.34). Thus, the solutions to the semi-discrete Thirring-like model (3.25), decaying as $n \to +\infty$, can be constructed from those of the linear vector differential-difference equation $\partial \boldsymbol{g}(n)/\partial\tau - \partial \boldsymbol{g}(n+1)/\partial\tau + \boldsymbol{g}(n) + \boldsymbol{g}(n+1) = \mathbf{0}$ through the compact formula

$$\boldsymbol{v}_n(\tau) = \boldsymbol{k}(n, n; \tau), \tag{3.35a}$$

$$\begin{aligned}\boldsymbol{k}(n,m) = \boldsymbol{g}(m) + \sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\big[\boldsymbol{k}(n,n+j) - \boldsymbol{k}(n,n+j+1)\big]C\big[\boldsymbol{g}(n+j+l) - \boldsymbol{g}(n+j+l+2)\big]^T \\ \times\big[\boldsymbol{g}(m+l) - \boldsymbol{g}(m+l+1)\big], \quad m \geq n.\end{aligned} \tag{3.35b}$$

Here, $\boldsymbol{v}_n = (v_n^{(1)}, v_n^{(2)}, \dots, v_n^{(M)})$. Using the formula (3.35), we can construct the soliton solutions of system (3.25) in a manner similar to that in the positive flow case. In particular, the “unrefined” one-soliton solution of (3.25) is given by

$$\boldsymbol{v}_n(\tau) = \frac{\boldsymbol{a}_1\mu_1^{-n}\mathrm{e}^{-\frac{\mu_1+1}{\mu_1-1}\tau} + \boldsymbol{a}_2\mu_2^{-n}\mathrm{e}^{-\frac{\mu_2+1}{\mu_2-1}\tau}}{1 - \frac{(\mu_1-\mu_2)(1-\mu_1)(1-\mu_2)}{(1-\mu_1\mu_2)^2}\langle \boldsymbol{a}_1 C, \boldsymbol{a}_2\rangle \mu_1^{-n}\mu_2^{-n}\mathrm{e}^{-\left(\frac{\mu_1+1}{\mu_1-1}+\frac{\mu_2+1}{\mu_2-1}\right)\tau}},$$

which can be rewritten in terms of the parametrization (3.32) as

$$\boldsymbol{v}_n(\tau) = \frac{\boldsymbol{b}_1\mathrm{e}^{\mathrm{i}\beta n - \mathrm{i}\frac{\sin\beta}{2\sinh\left(\frac{\alpha+\mathrm{i}\beta}{2}\right)\sinh\left(\frac{\alpha-\mathrm{i}\beta}{2}\right)}\tau} + \boldsymbol{b}_2\mathrm{e}^{-\mathrm{i}\beta n + \mathrm{i}\frac{\sin\beta}{2\sinh\left(\frac{\alpha+\mathrm{i}\beta}{2}\right)\sinh\left(\frac{\alpha-\mathrm{i}\beta}{2}\right)}\tau}}{\cosh\left[\alpha n + \frac{\sinh\alpha}{2\sinh\left(\frac{\alpha+\mathrm{i}\beta}{2}\right)\sinh\left(\frac{\alpha-\mathrm{i}\beta}{2}\right)}\tau + \delta\right]},$$

with the condition $4\mathrm{i}(\cosh\alpha - \cos\beta)\sin\beta\,\langle \boldsymbol{b}_1 C, \boldsymbol{b}_2\rangle = (\sinh\alpha)^2$. This provides the bright one-soliton solution under the “reality conditions” $\alpha > 0$, $0 < \beta < \pi$ (or $-\pi < \beta < 0$), and $\mathrm{e}^{2\delta} \notin \mathbb{R}_{<0}$, which is indeed regular for real n and τ.

4 Concluding remarks

In this paper, we have proposed a new type of reduction involving an antisymmetric constant matrix; this reduction relates one matrix variable with another in a system of coupled matrix PDEs. In the particular case of vector variables, it enables us to obtain the integrable vector PDEs having $Sp(m)$ as their symmetry group. Our approach has been proven to be applicable to both continuous and discrete systems. For the particularly interesting systems such as (2.7), (2.22), (2.34), (3.7), (3.20), and (3.25), the one-soliton solutions are derived from the (discrete) linear integral equations of the Gel’fand–Levitan–Marchenko type. The solutions are clearly invariant up to a redefinition of the soliton parameters under the same symmetry group $Sp(m)$ as that of the original systems. The behavior of the solitons reflects an interesting characteristic of these systems; the total “particle number” of the system, *e.g.*, $\int_{-\infty}^{\infty}\|\boldsymbol{u}\|^2\mathrm{d}x$ in the continuous case, is, in general, not conserved and varies in time. As a result, each soliton exhibits an overall vectorial

oscillation, in addition to an internal oscillation among the vector components. A detailed investigation of the multi-soliton solutions would be an interesting and promising path toward the construction of a new class of set-theoretical solutions with symplectic invariance to the quantum Yang–Baxter equation (cf. ref. 45).

One natural question arising from the results of subsections 2.1 and 2.2 concerns the integrability of a system of the following general form:

$$\frac{\partial u_i}{\partial t} + \frac{\partial^3 u_i}{\partial x^3} + a\left[\sum_{1\le j<k\le M} C_{jk}\Big(\frac{\partial u_j}{\partial x}u_k - u_j\frac{\partial u_k}{\partial x}\Big)\right]\frac{\partial u_i}{\partial x} + b\frac{\partial}{\partial x}\left[\sum_{1\le j<k\le M} C_{jk}\Big(\frac{\partial u_j}{\partial x}u_k - u_j\frac{\partial u_k}{\partial x}\Big)u_i\right] = 0, \quad i = 1, 2, \ldots, M, \tag{4.1}$$

where a and b are constants. Using the *Mathematica* package "InvariantsSymmetries.m" [46], we searched for the cases wherein system (4.1) with $M = 2$ can possess higher polynomial conservation laws and/or symmetries of the prescribed orders. The result was null in that within the limitations of our computer's memory and CPU performance, we could detect no integrable case except the already found two cases $a = 0$ and $b = 0$. Our future task is to rigorously prove (or disprove) the nonintegrability of system (4.1) in the case $ab \neq 0$.

We would like to explain how the semi-discrete systems of the form $\partial_t u_n^{(i)} + (u_{n+1}^{(i)} - u_{n-1}^{(i)}) + \{\text{nonlinear terms}\} = 0$ presented in section 3 can be related to the third-order PDEs of the form $\partial_T u_i + \partial_X^3 u_i + \{\text{nonlinear terms}\} = 0$ in a continuous limit. In fact, the asymptotic expansion with respect to the space interval $\varDelta$, $\partial_t u_n^{(i)} + (u_{n+1}^{(i)} - u_{n-1}^{(i)}) \simeq \partial_t u^{(i)} + 2\varDelta\partial_x u^{(i)} + \frac{1}{3}\varDelta^3\partial_x^3 u^{(i)} + O(\varDelta^5)$, implies that a Galilean plus scaling transformation such as $\partial_t + 2\varDelta\partial_x =: \frac{1}{3}\varDelta^3\partial_T$, $\partial_x =: \partial_X$, $u^{(i)} \sim O(\varDelta^{\frac{1}{2}})$, or equivalently, $u_n^{(i)}(t) \sim \varDelta^{\frac{1}{2}} u_i\big(\varDelta(n-2t), \varDelta^3 t/3\big)$ must be performed. This is a commonly accepted technique (see, *e.g.*, refs. 43, 47), and the solutions of such a semi-discrete system generally have the same structures as those of the corresponding continuous system. However, one can also obtain further "natural" space discretizations that directly arrive at the third-order PDEs in the continuum limit, without resorting to the Galilean transformations. This is achieved by considering a proper linear combination of the original semi-discrete system and a higher symmetry of it [47, 48]. Note that this "improvement" results only in a minor change in the time dependence of certain parameters in the solutions, while a semi-discrete system obtained in this manner usually appears rather complicated and less attractive than the original one. Therefore, we do not pursue such a direction in this paper.

We have concentrated on the reductions of the third-order ($\omega \propto k^3$) flows of the derivative NLS (DNLS)-type hierarchies as well as their first negative ($\omega \propto k^{-1}$) flows, in both the continuous and semi-discrete cases. The feasibility of the reduction is based on the fact that the cubic terms in the evolution equation for q and those for r in such systems, *e.g.*, (2.2) or (2.19), have opposite signs. This is in contrast with the case of the corresponding flows of the matrix NLS hierarchy, *e.g.*, the (non-reduced) matrix complex mKdV equation (2.10) that permits various reductions, including $R = A_1 Q^T A_2$ with $A_1^T = A_1$ and $A_2^T = A_2$ or $A_1^T = -A_1$ and $A_2^T = -A_2$, but not the reduction $R = CQ^T$ or $R = Q^T C$ with $C^T = -C$. However, it should be noted that the matrix DNLS flows are not the only class of systems that permit the reductions of the latter type exploited in

this paper. As an illustrative example, let us consider the matrix generalization of the Yajima–Oikawa system [49] (cf. refs. 50–55),

$$\begin{cases} \mathrm{i}Q_{t_2} + Q_{xx} - PQ = O, \\ \mathrm{i}R_{t_2} - R_{xx} + RP = O, \\ \mathrm{i}P_{t_2} + 2(QR)_x = O, \end{cases} \tag{4.2}$$

and its third-order symmetry. System (4.2) possesses a Lax pair of the form

$$U = \mathrm{i}\zeta \begin{bmatrix} -I_1 & & \\ & O & \\ & & I_1 \end{bmatrix} + \begin{bmatrix} O & Q & P \\ O & O & R \\ I_1 & O & O \end{bmatrix}, \tag{4.3a}$$

$$V = \mathrm{i}\zeta^2 \begin{bmatrix} O & & \\ & I_2 & \\ & & O \end{bmatrix} + \zeta \begin{bmatrix} O & Q & O \\ O & O & -R \\ O & O & O \end{bmatrix} + \mathrm{i} \begin{bmatrix} O & Q_x & QR \\ R & O & -R_x \\ O & Q & O \end{bmatrix}. \tag{4.3b}$$

This implies that the substitution of (4.3) in the zero-curvature condition (2.4) yields (4.2). The matrix Yajima–Oikawa system (4.2) allows the standard reduction $R = BQ^\dagger$, $P^\dagger = P$, $B^\dagger = -B$, $B_t = B_x = O$. The third-order symmetry of (4.2) reads as

$$\begin{cases} Q_{t_3} + 4Q_{xxx} - 3P_x Q - 6PQ_x - 6QRQ = O, \\ R_{t_3} + 4R_{xxx} - 3RP_x - 6R_x P + 6RQR = O, \\ P_{t_3} + P_{xxx} - 3(P^2)_x + 6(Q_x R - QR_x)_x = O, \end{cases} \tag{4.4}$$

and the corresponding Lax pair is given by (4.3a) and

$$\begin{aligned} V = \mathrm{i}\zeta^3 &\begin{bmatrix} -4I_1 & & \\ & O & \\ & & 4I_1 \end{bmatrix} + \zeta^2 \begin{bmatrix} O & 4Q & 4P \\ O & O & 4R \\ 4I_1 & O & O \end{bmatrix} + \mathrm{i}\zeta \begin{bmatrix} -2P & 4Q_x & 2P_x \\ O & O & 4R_x \\ O & O & 2P \end{bmatrix} \\ &+ \begin{bmatrix} P_x + 2QR & -4Q_{xx} + 2PQ & -P_{xx} + 2P^2 + 2(QR_x - Q_x R) \\ 4R_x & -4RQ & -4R_{xx} + 2RP \\ 2P & -4Q_x & -P_x + 2QR \end{bmatrix}. \end{aligned}$$

System (4.4) allows an extension of the typical reduction considered in this paper, that is, $R = CQ^T$ and $P^T = P$ or $R = Q^T C$ and $P^T C = CP$, where C is an antisymmetric constant matrix. In particular, the vector reduction in the former case changes the matrix system (4.4) into the following system [51, 56]:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + 4\dfrac{\partial^3 u_i}{\partial x^3} - 3\dfrac{\partial p}{\partial x} u_i - 6p\dfrac{\partial u_i}{\partial x} = 0, \quad i = 1, 2, \dots, M, \\ \dfrac{\partial p}{\partial t} + \dfrac{\partial^3 p}{\partial x^3} - 6p\dfrac{\partial p}{\partial x} + 12 \displaystyle\sum_{1 \le j < k \le M} C_{jk} \Big(\dfrac{\partial^2 u_j}{\partial x^2} u_k - u_j \dfrac{\partial^2 u_k}{\partial x^2} \Big) = 0. \end{cases}$$

It is noted that this system is a modification of the triangular system comprising the KdV equation and time part of the associated linear problem due to the addition of the last summation term.

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